pf of claim #2°:

each $i \in \{1,\cdots,k\}$ generates $\xi_i$ offsprings

$$E(X_n) = \sum_{k=0}^{\infty} \underbrace{E(X_n \mid X_{n-1}=k)}_{\xi_1+\cdots+\xi_k} P(X_{n-1}=k)$$

$$= E(\xi_1+\cdots+\xi_k)$$

$$= k E(\xi)$$

$$= k\mu$$

$$= \mu \sum_{k=0}^{\infty} k P(X_{n-1}=k)$$

$$= \mu E(X_{n-1})$$

$$\therefore \quad E(X_n) = \mu E(X_{n-1}) = \cdots = \mu^n E(X_0)$$

#

What if $\mu \geq 1$ ?

Indeed, we have more general strategy to treat all cases of $\mu$:

extinction prob.

$$\rho = \rho_{10} = P_1(T_0 < \infty)$$

$$= p_0 + \sum_{k=1}^{\infty} p_k \underbrace{\rho_{k0}}_{\overset{\text{claim}}{=}\ \rho^k}$$

due to indepence

$$= \sum_{k=0}^{\infty} p_k \, \rho^k \quad \leftarrow (\rho^0 = 1)$$

$\therefore$ $\rho$ is the solution to equation

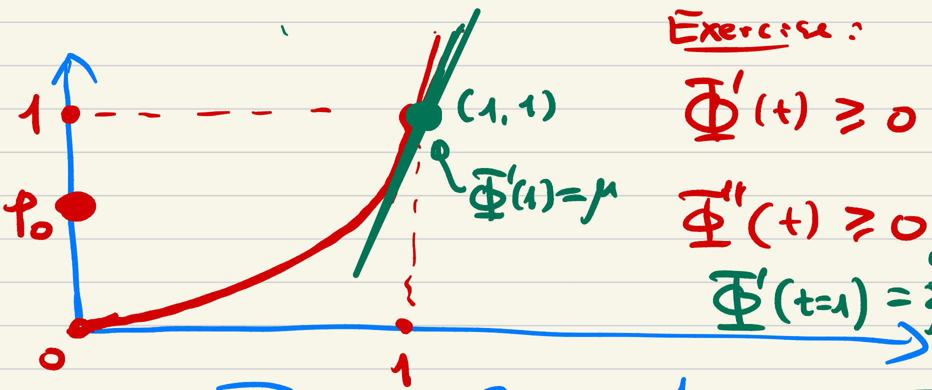
$$t = \Phi(t) \overset{\text{def.}}{=} \sum_{k=0}^{\infty} p_k \, t^k$$

called moment generating function of the pdf $(p_k)_{k=0}^{\infty}$ for the r.v. $\xi$.

Observe :

$\Phi(0) = p_0 \in (0,1)$

$\Phi(1) = \sum_{k=0}^{\infty} p_k = 1$




Exercise :

$\Phi'(t) \geq 0$

$\Phi''(t) \geq 0$

$$\Phi'(t=1) = \sum_{k=0}^{\infty} k \, p_k = E(\xi) = \mu$$

$$\Phi(t) = \sum_{k=0}^{\infty} p_k \, t^k = p_0 + p_1 t + p_2 t^2 + \cdots$$

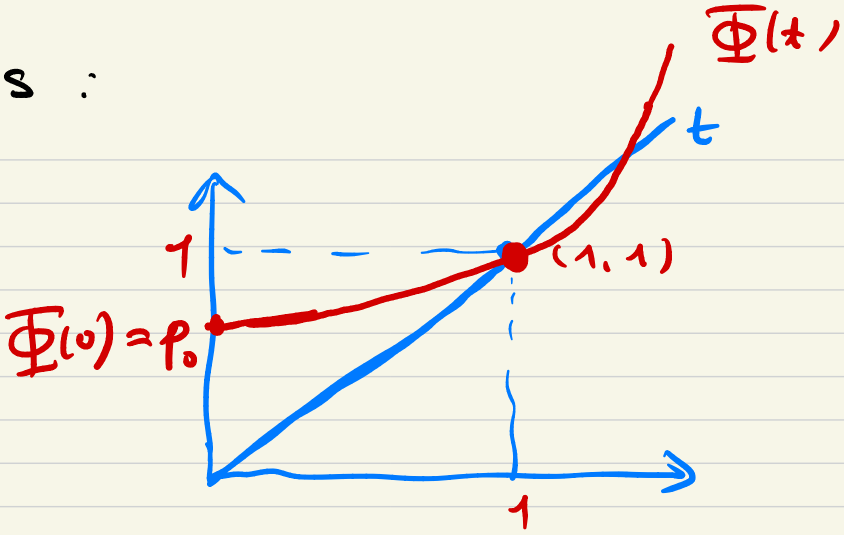
Consider possible solutions to

$$t = \Phi(t), \qquad t \in [0, 1] \; ?$$

# Three cases:

## Case 1.

$E(\xi) = \mu < 1$

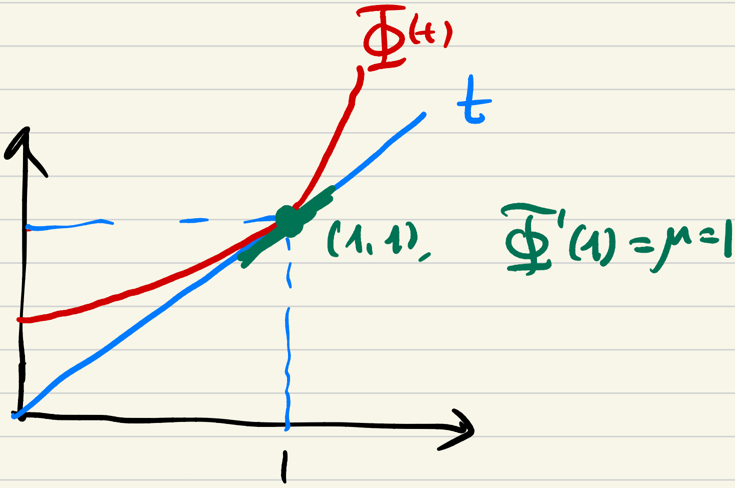



Exactly one solution

$\therefore \rho = 1$

## Case 2

$E(\xi) = \mu = 1$

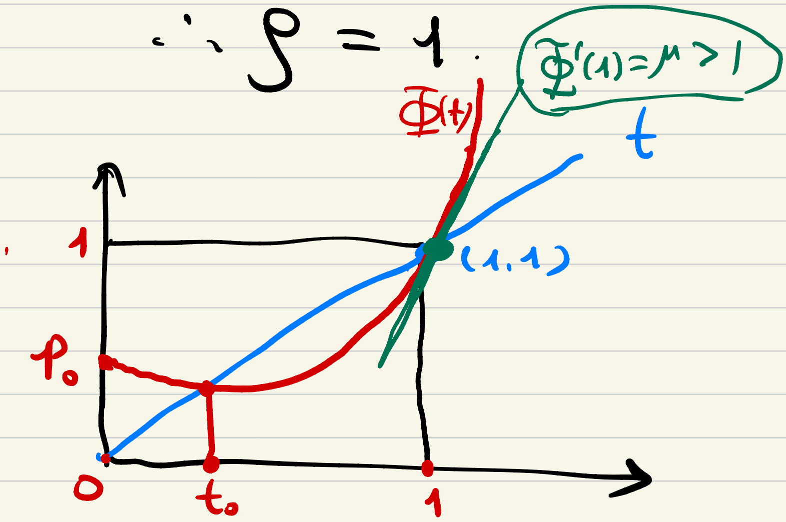



Exactly one solution

$\therefore \rho = 1$.

## Case 3

$E(\xi) = \mu > 1$




Two solutions to $t = \Phi(t)$

either $\underbrace{t_0}_{\in (0,1)}$ or $1$.

$\boxed{t_0 = \Phi(t_0)}$

In this case, we can disregard the solution 1

**Claim** $\rho \leq t_0$ $\quad$ ($\therefore$ $\rho = t_0$)

**pf.** $\rho = P_1(T_0 < \infty)$

$$= \lim_{n\to\infty} \underbrace{P_1(T_0 \leq n)}_{\overset{\text{def}}{=} a_n} \boxed{\leq t_0}$$

to show (by induction):

$a_n \leq t_0, \quad n = 0, 1, \ldots$

Indeed,

$n=0$: $\quad a_0 = P_1(T_0 \leq 0) = 0 < t_0$

note: "$T_0 \geq 1$"

Assume "TRUE" for $n \geq 0$, to show "TRUE" for $n+1$

$$a_{n+1} = P_1(T_0 \leq n+1)$$

Exercise in Homework

$$= \underbrace{P(1,0)}_{=p_0} + \sum_{k=1}^{\infty} \underbrace{P(1,k)}_{=p_k} \underbrace{P_k(T_0 \leq n)}_{=a_n^k}$$

$a_n = P_1(T_0 \leq n)$

$$= \sum_{k=0}^{\infty} p_k a_n^k \qquad (a_n^0 = 1)$$

I.A.: $a_n \leq t_0$

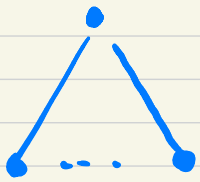
$$= \Phi(a_n) \underset{\Phi' \geq 0}{\leq} \Phi(t_0) = t_0$$

$\therefore\ a_{n+1} \le t_0$, TRUE for $n+1$.

$\#$

## Recall:

### Branching chain



$\xi$ = no of offsprings that one individual generates in the previous stage

pdf of $\xi$: $(p_k)_{k\ge 0}$

$\downarrow$

$$\Phi(t) = \sum_{k=0}^{\infty} p_k t^k$$

moment function generated by $(p_k)_{k\ge 0}$

$$\mu \overset{\text{def.}}{=} E(\xi)$$

**Topic:** extinction prob

$$\rho \overset{\text{def.}}{=} \rho_{10}$$

① two trivial cases

$p_0 = 0 \longrightarrow$

$p_0 = 1 \longrightarrow$

② non-trivial case

$p_0 \in (0,1)$

$\mu \le 1 \Rightarrow \rho = 1$

$\mu > 1 \Rightarrow$ Two solns to $\Phi(t) = t$

$t_0 \in (0,1)$ or $1$

discard

$\rho = t_0 \in (0,1)$

e.g. Each family has 3 kids

with $\frac{1}{2}$ for boy

$\frac{1}{2}$ for girl

Find the prob. that the male line eventually extinct.

<u>Sol</u>: $\xi \stackrel{\text{def}}{=}$ no of boys generated

$(p_k)_{k=0}^{\infty}$ —— $p_k = 0$ for $k \geq 4$

$$\left.\begin{aligned} p_0 &= P(\underset{\text{no boy}}{\underset{\uparrow}{\xi=0}}) = \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8} \\ p_1 &= P(\underset{\text{one boy}}{\underset{\uparrow}{\xi=1}}) = 3\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^2 = \tfrac{3}{8} \\ p_2 &= P(\underset{\text{2 boys}}{\underset{\uparrow}{\xi=2}}) = \cdots = \tfrac{3}{8} \\ p_3 &= P(\underset{\text{3 boys}}{\underset{\uparrow}{\xi=3}}) = \tfrac{1}{8} \end{aligned}\right\}$$

$$E(\xi) = \sum_{k=0}^{\infty} k p_k = \sum_{k=0}^{3} k p_k = \cdots = \frac{3}{2} > 1$$

$$\Phi(t) = \sum_{k=0}^{\infty} p_k t^k = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

$$= \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3$$

Consider :

$$\Phi(t) = t, \quad \text{i.e.} \quad \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3 = t$$

$$\cdots \Rightarrow (t-1)\left(\tfrac{1}{8}t^2 + \cdots\right) = 0$$

$$\therefore \quad \underbrace{t-1=0}_{\text{discard}} \quad \text{or} \quad \tfrac{1}{8}t^2 + \cdots = 0$$

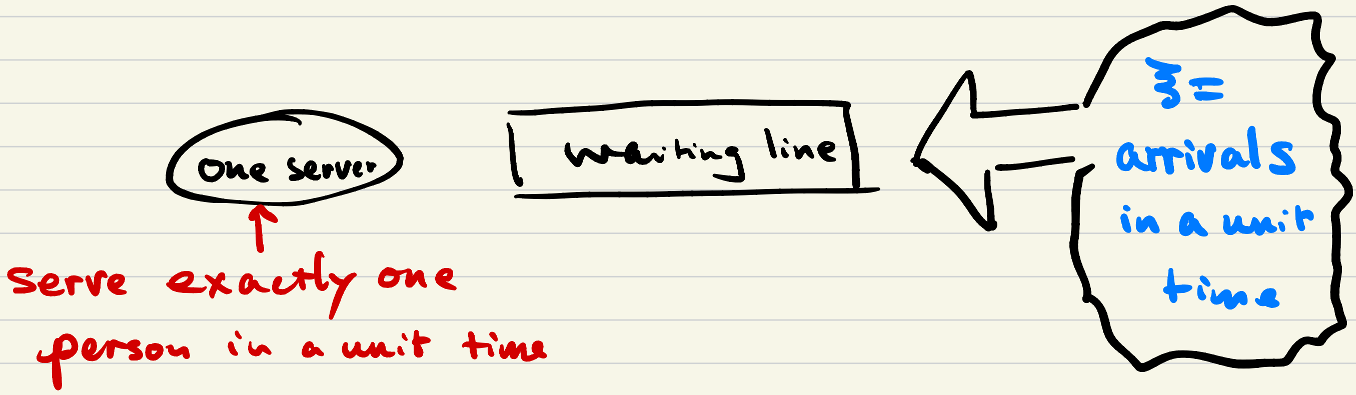
$$\therefore \quad t = \sqrt{5} - 2$$

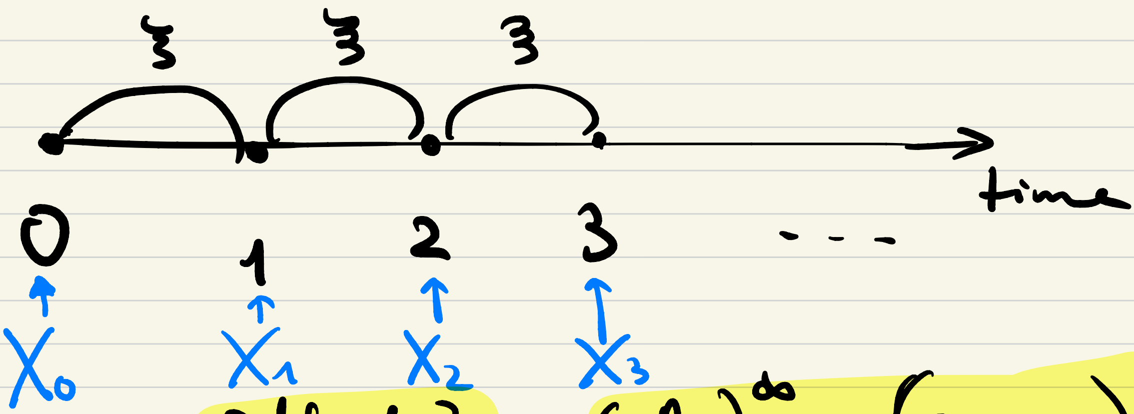
$\therefore$ extinction prob. for line of male population

is $\quad \rho = \sqrt{5} - 2 \in (0, 1)$. #

## Type # 3. Queuing chain

Recall the setting of MC :




pdf of $\xi$ : $\quad (p_k)_{k=0}^{\infty} = \left(P(\xi = k)\right)_{k=0}^{\infty}$

$$\text{MC}: \{X_n\}_{n=0}^{\infty}$$

$X_n \stackrel{\text{def.}}{=}$ no. of persons on the line waiting for service at the time $n$.

State space : $S = \{0, 1, \cdots\}$

Transition prob. :

$$P(0,y) \stackrel{y\geq 0}{=} P(X_1 = y \mid X_0 = 0) = P(\xi = y) = f_y$$

$X_1 = \xi$

$X_1 = \underset{\geq 1}{X_0} + \xi - 1 \Rightarrow y = x + \xi - 1 \Rightarrow \xi = y - x + 1$

$$P(x,y) \underset{y\geq 0}{\overset{x\geq 1}{=}} (\cdots) = P(\xi = y - x + 1)$$

$$= \begin{cases} f_{y-x+1} & \text{if } y \geq x-1 \\ 0 & \text{if } 0 \leq y < x-1 \end{cases}$$

Observation :

$$P(0,y) = f_y = P(1,y), \quad \forall y \geq 0$$

General question :

Assume chain is irreducible

then is it recurrent or transient?
$(\Leftrightarrow \rho = 1)$ $(\Leftrightarrow \rho < 1)$

$$\rho \stackrel{\text{def.}}{=} \rho_{00} = P_0(T_0 < \infty)$$

**Thm**: $\rho$ is a solution to

$$t = \Phi(t) \overset{\text{def.}}{=} \sum_{k=0}^{\infty} p_k t^k$$

↑ moment generating function of $(p_k)_{k\geq 0}$

If agree, then similar before,

- $\mu = E(\xi) \leq 1$ : $t = \Phi(t)$ has a unique solution $t=1$

  $\therefore$ $\underline{\underline{\rho = 1}}$ recurrent

- $\mu = E(\xi) > 1$ : $t = \Phi(t)$ has two solutions $t_0 \in (0,1)$ , $1$ (discard)

  $\therefore$ $\rho = \underline{\underline{t_0}} \in (0,1)$ transient

Proof of Thm :

$$\rho = \rho_{00}$$

$$= P_0(T_0 < \infty)$$

$$= \underbrace{P(\underline{\underline{0}}, \underline{\underline{0}})} + \boxed{\sum_{k \neq 0}} \underbrace{P(0, k)} \boxed{\rho_{k0}}$$

$\sum_{k=1}^{\infty}$ ?

$P_0$ $\qquad\qquad$ $P_k$

it suffices to show:

$$\rho_{k0} = \rho^k, \quad k = 1, 2, \cdots$$

Consequence of three claims

$$\begin{cases} \text{Claim \#1}: & \rho_{10} = \rho \\ \text{Claim \#2}: & \rho_{k,0} = \rho_{k,k-1}\, \rho_{k-1,0}, \quad k=2, \cdots \\ \text{Claim \#3}: & \rho_{k,k-1} = \rho_{10}, \quad k=2, \cdots \end{cases}$$

if so, then

$$\rho_{k0} \overset{\text{claim 2}}{=\!=} \rho_{k,k-1}\, \rho_{k-1,0} \overset{\text{claim 3 + claim \#1}}{=\!=} \rho\, \rho_{k-1,0}$$

$$= \cdots = \rho^{k-1} \rho_{10} = \rho^{k-1} \rho = \rho^k$$

#

Proof of claim# 1:

$$\rho_{00} \overset{\text{ok}}{\underset{?}{\neq}} \rho_{10}$$

$$\rho_{10} = P(1,0) + \sum_{k \neq 0} P(1,k)\, \rho_{k0}$$

$$\rho_{00} = P(0,0) + \sum_{k \neq 0} P(0,k)\, \rho_{k0}$$

#

Proof of claim #3: $\left(\rho_{k,k-1} = \rho_{1,0},\quad k=2,3,\cdots\right)$

$$\rho_{k,k-1} = P_k\left(\underbrace{1 \le T_{k-1} < \infty}_{=\bigcup_{n=1}^{\infty}\{T_{k-1}=n\}}\right) \qquad (k\geq 2)$$

disjoint union

$$= \sum_{n=1}^{\infty} \underbrace{P_k(T_{k-1}=n)}_{=P(T_{k-1}=n \mid X_0=k)}$$

Consider $n\geq 1$:

$$\boxed{\text{“}T_{k-1}=\underset{\geq 1}{n}\text{” given that } X_0 = \underset{\geq 2}{k}}$$

i.e.

$$n = \min\left\{ m\geq 1 : \underbrace{\cancel{k} + (\xi_1-1)+\cdots+(\xi_m-1) = \cancel{k}-1}\right\}$$

$$\Updownarrow$$

$$n = \min\left\{ m\geq 1 : 1 + (\xi_1-1)+\cdots+(\xi_m-1) = 0 \right\}$$

i.e. $\boxed{T_0 = n \text{ given } X_0 = 1}$

this shows:

$$P_k(T_{k-1}=n) \overset{\forall n\geq 1}{=} \underbrace{P_1(T_0=n)}_{\text{indep't of } k}$$

$\forall\, k=2,\ldots$

Plug it back.

$$\therefore \; \rho_{k,k-1} = \sum_{n=1}^{\infty} P_1(T_0 = n)$$

$$= P_1(1 \le T_0 < \infty)$$

$$= \rho_{10}, \quad \forall k = 2, \cdots$$

#

Proof of claim#2: i.e.

to show: $\rho_{k,0} = \rho_{k,k-1}\, \rho_{k-1,0}$, $k = 2, \cdots$

Indeed,

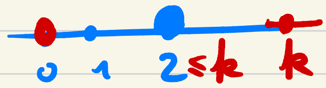
$$\rho_{k,0} \overset{k \ge 2}{=} P_k(1 \le T_0 < \infty)$$

$$= \sum_{m=1}^{\infty} P_k(T_0 = m)$$

Term of $m=1$ vanishes: $P_{\underset{\ge 2}{k}}(T_0 = 1) = 0$

$$= \sum_{m=2}^{\infty} P_{\underset{\ge 2}{k}}(T_0 = \underset{\ge 2}{m})$$

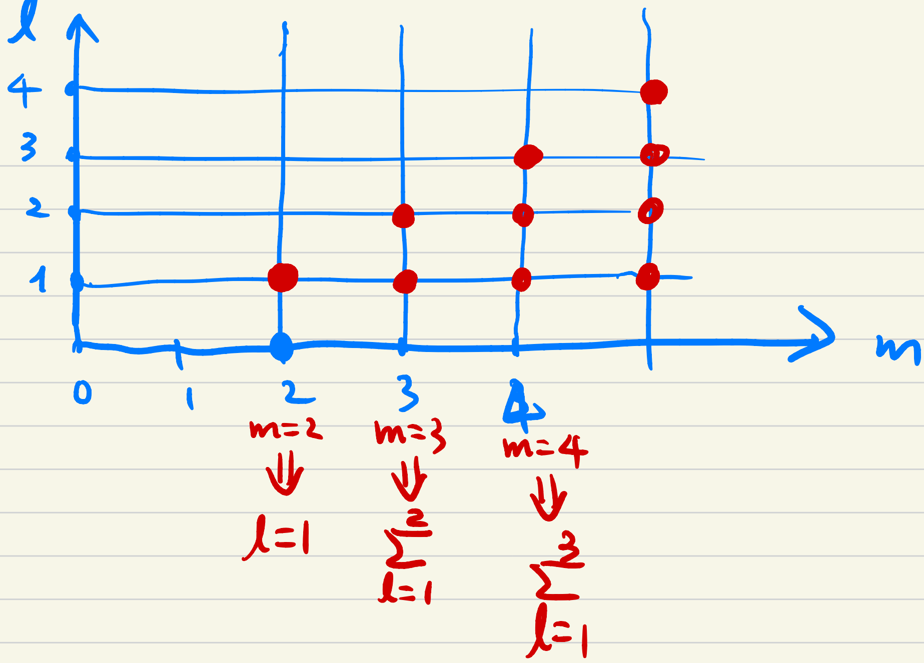
$\Rightarrow k-1 \ge 1$




$T_0 \ge 2$

$$= \boxed{\sum_{m=2}^{\infty} \sum_{l=1}^{m-1}} P_k(\underset{\ge 1}{T_{k-1}} = l)\, P_{k-1}(\underset{\ge 1}{T_0} = m-l)$$

$$= \sum_{l=1}^{\infty} \sum_{m=l+1}^{\infty} [\cdots\cdots]$$




$$= \sum_{l=1}^{\infty} P_k(T_{k-1}=l)\left[\sum_{m=l+1}^{\infty} P_{k-1}(T_0=m-l)\right]$$

$l \geq 1$ fixed

$m-l=1$

$n \overset{\text{def.}}{=} m-l$

$$= \sum_{n=1}^{\infty} P_{k-1}(T_0=n)$$

$$= P_{k-1}(1 \leq T_0 < \infty)$$

$$= \rho_{k-1,0}$$

$$= \rho_{k-1,0} \sum_{l=1}^{\infty} P_k(T_{k-1}=l)$$

$$= \rho_{k-1,0}\, P_k(1 \leq T_{k-1} < \infty)$$

$$= \rho_{k-1,0}\, \rho_{k,k-1} \qquad \#$$