Next goal:

Decomposition of state space

Def. We say $x \to y$ (x leads to y) if $\rho_{xy} > 0$.

Lemma: (i) $x \to y$ iff $p^n(x,y) > 0$ for some $n \geq 1$.

pf. $\rho_{xy} = P_x(1 \leq T_y < \infty) > 0$

$\Leftrightarrow p^n(x,y) > 0$, for some $n \geq 1$.

(ii) If $x \to y$ & $y \to z$, then $x \to z$.

pf: $x \to y \Rightarrow \exists\, n \geq 1$, s.t. $p^n(x,y) > 0$

$y \to z \Rightarrow \exists\, k \geq 1$, s.t. $p^k(y,z) > 0$

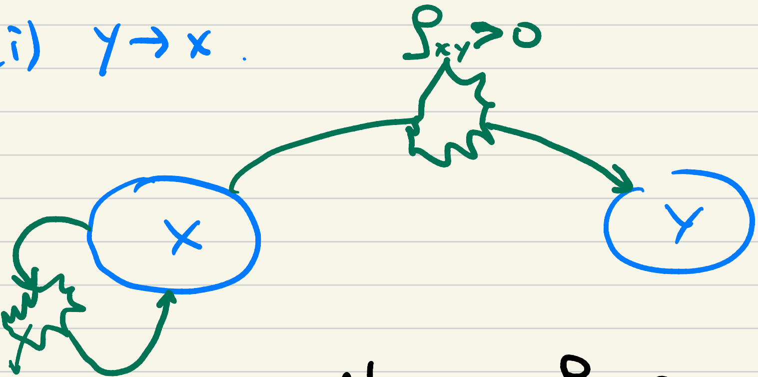
$$\underbrace{p^{n+k}}_{= p^n \cdot p^k}(x,z) \geq \underbrace{p^n(x,y)}_{>0}\, \underbrace{p^k(y,z)}_{>0} > 0$$

$\therefore x \to z$. #

Prop.

$$\left.\begin{array}{l} x \text{ recurrent} \\ (\rho_{xx}=1) \\ x \to y \\ (\rho_{xy}>0) \end{array}\right\} \Rightarrow \begin{array}{ll} \text{(i)} & y \to x \\ \text{(ii)} & y \text{ also recurrent} \\ \text{(iii)} & \rho_{xy} = 1 = \rho_{yx} \end{array}$$

pf. (i) $y \to x$.



$\rho_{xx}=1$

$\Updownarrow$

$P_x(T_x=\infty)=0$

claim: $\rho_{yx}>0$.

otherwise: $\rho_{yx}=0$

contradiction to " $\rho_{xx}=1$ "

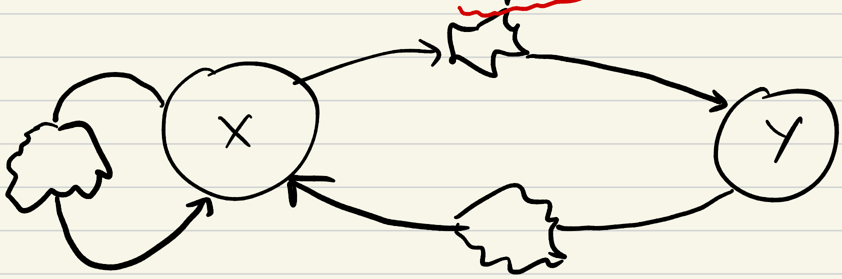
$\rho_{yx}=0 \Leftrightarrow P_y(T_x=\infty)=1$

(ii) y is recurrent.

Heuristic understanding:

$\rho_{xy}>0 \Rightarrow \exists\, n_0 \geq 1$, s.t. $p^{n_0}(x,y)>0$

$\rho_{xx}=1$



? y recurrent

$\Leftrightarrow E_y(N(y))=\infty$

$E_x(N(x))=\infty$

$\rho_{yx}>0 \Rightarrow \exists\, n_1 \geq 1$, s.t. $p^{n_1}(y,x)>0$

Mathematical proof :

$$p^1(y,y) + p^2(y,y) + \cdots$$

part of the above

$$p^{n_1+1+n_0}(y,y) + p^{n_1+n_0+2}$$

$$\infty \overset{\text{ok}}{=} E_y(N(y)) = \sum_{n=1}^{\infty} p^n(y,y)$$

$$\geq \sum_{k=1}^{\infty} \underbrace{p^{n_1+k+n_0}(y,y)}$$

$$= (p^{n_1} \cdot p^k \cdot p^{n_0})(y,y)$$

$$\geq p^{n_1}(y,x)\, p^k(x,x)\, p^{n_0}(x,y)$$

(x,y)-entry of Product of p^{m_1}, p^{m_2} & p^{m_3}

$$p^{m_1+m_2+m_3}(x,y)$$

$$= \sum_{x_1,x_2\in S} p^{m_1}(x,x_1)\, p^{m_2}(x_1,x_2)\, p^{m_3}(x_2,y)$$

$$\geq \underbrace{p^{n_1}(y,x)}_{\substack{\text{finite } (\leq 1)\\ >0}} \underbrace{\left(\sum_{k=1}^{\infty} p^k(x,x)\right)}_{= E_x(N(x))} \underbrace{p^{n_0}(x,y)}_{\substack{\text{finite } (\leq 1)\\ >0}}$$

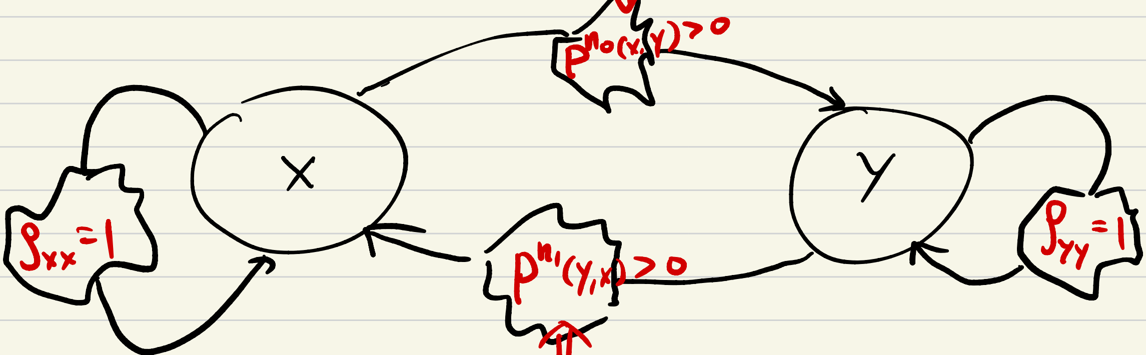
$= \infty$ ($\because$ x is recurrent)

$$= \infty .$$

$\therefore$ y is recurrent.

(iii) $\rho_{yx} = 1$. Indeed $\rho_{xy} > 0$

$\Downarrow$



$\Uparrow$

$\rho_{yx} > 0$

otherwise, $\rho_{yx} < 1$,

i.e. $\underbrace{1 - \rho_{yx}}$ > 0

$$= P_y(T_x = \infty)$$

with this (+)-prob. the chain from y will Never visit x. $\Rightarrow P_x(T_x = \infty) > 0$

exchang x and y similarly $\rho_{xy} = 1$ holds true.

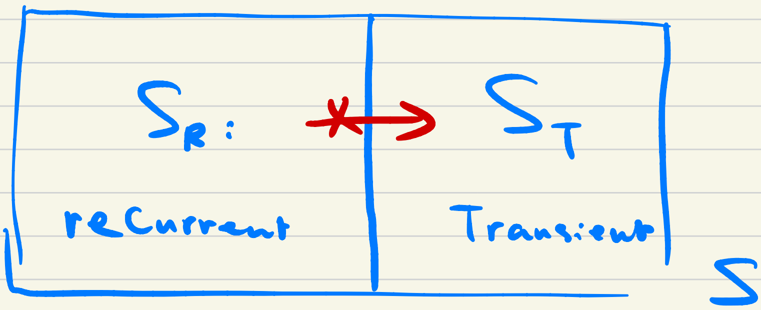
Contradiction to "x is recurrent"

#

$$S = S_R \cup S_T$$

S: state space

$\cup$: disjoint union



no recurrent state can lead to any transient state

$\therefore$ $S_R = \{\text{all recurrent states}\}$ is "closed".

Def. $C \subseteq S$ is closed if

$$\rho_{xy} = 0, \quad \forall x \in C, \ \forall y \notin C$$

$(x \not\to y)$

i.e. no state inside C leads to any state outside C.

Remarks:

① The following statements are equivalent.

(a) C is closed $\quad$ ($\rho_{xy} = 0, \ \forall x \in C, \ \forall y \notin C$)

(b) $P^n(x,y) = 0, \ \forall x \in C, \ \forall y \notin C, \ \forall n \geq 1,$

(c) $P(x,y) = 0, \ \forall x \in C, \ \forall y \notin C.$

Pf. direct to see: $\quad$ (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c)

it suffices to show: $\quad$ (c) $\Rightarrow$ (b)

Fix $x \in C$ and $y \notin C$,

$$
\begin{aligned}
P^2(x,y) &= \sum_{x_1 \in S} P(x,x_1)\, P(x,y) \\
&= \sum_{\substack{x_1 \in S \\ x_1 \in C}} P(x,x_1)\, \cancel{\underbrace{P(\overset{\in C}{x_1}, \overset{\notin C}{y})}} \\
&\quad + \sum_{\substack{x_1 \in S \\ x_1 \notin C}} \cancel{\underbrace{P(\overset{\in C}{x}, \overset{\notin C}{x_1})}}\, P(x_1, y) \\
&= 0 + 0 = 0
\end{aligned}
$$

repeatedly use the same argument

$$P^n(x,y) = 0, \qquad \forall n \geq 2.$$

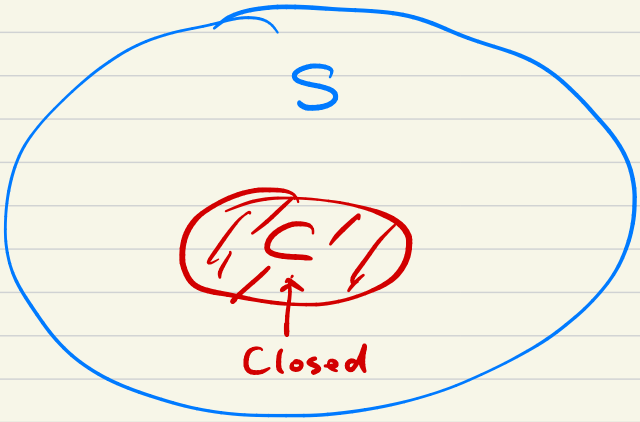
② If C is closed, $x \in C$, and $P(x,y) > 0$ then $y \in C$.

Pf. Contradiction + (c) in ①.

③ If $C \subseteq S$ is closed, then a MC

$$\{X_n\}_{n=0}^{\infty}$$

over S can also be regarded as a Markov chain restricted to the state space C.



$X_0, X_1, X_2, \ldots$

$X_0(\Omega) \subset C$, i.e. $X_{0,\omega} \in C$

Def. ① A closed set C is irreducible if

$$x \to y, \quad \forall x, y \in C$$

i.e. any two states in C can lead to eacher other.

② A Markov chain $\{X_n\}_{n=0}^{\infty}$ is irreducible if the state space S is irreducible.

Continue : $S = S_R \cup S_T$

closed set

disjoint

1°. $S_R \neq \phi$,

then $\exists\, x_1$ recurrent $\in S_R$

$$C_{x_1} \overset{\text{def.}}{=} \{ x \in S_R : \overset{\text{recurrent}}{x_1} \to x \} \neq \phi$$

(contains x_1)

Claim : C_{x_1} is closed and irreducible

pf. : 1°. "Closed" : $\left.\begin{array}{l} x \in C_{x_1} \\ x \to y \end{array}\right\} \Rightarrow y \in C_{x_1}$

Indeed, $\overset{\text{recurrent}}{x_1} \to x \to \overset{\text{recurrent}}{y}$

$\therefore\ y \in C_{x_1}$

2°. "Irreducible" :

$$\left.\begin{array}{l} x \in C_{x_1} \\ y \in C_{x_1} \end{array}\right\} \Rightarrow x \to y$$

Indeed, $x \in C_{x_1} \Rightarrow \overset{\text{recurrent}}{x_1} \rightleftarrows \overset{\text{recurrent}}{x}$

$y \in C_{x_1} \Rightarrow x_1 \longrightarrow y$

$\therefore\ \overset{\text{recurrent}}{x} \longrightarrow x_1 \longrightarrow \overset{\text{recurrent}}{y}$

$x \to y$. #

Rk: C_{x_1} is the "largest" closed & irreducible set containing x_1.

2°. $S_R \setminus C_{x_1} = \phi$ or $\neq \phi$

If $S_R \setminus C_{x_1} \neq \phi$

then $\exists\, x_2 \in S_R \setminus C_{x_1}$ (i.e. $x_2 \in S_R$, $x_2 \notin C_{x_1}$)

define again, $C_{x_2} \overset{\text{def.}}{=} \{ x \in S_R : x_2 \to x \}$

is closed and irreducible.

Claim: $C_{x_2} \cap C_{x_1} = \phi$.

Pf: Exercise. (use contradiction)

$\Downarrow$

$C_{x_1} = C_{x_2}$

$\Downarrow$

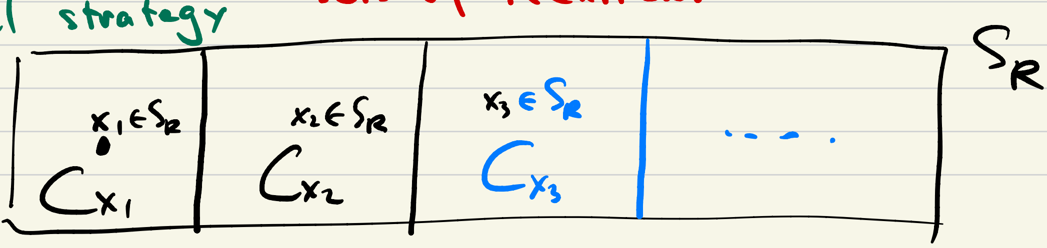
contradiction to $x_2 \notin C_{x_1}$.

3°. Continue the process:

$$S_R \setminus \Big(\underbrace{\bigcup_{i=1}^{2} C_{x_i}}_{} \Big) = \phi \text{ or } \neq \phi$$

disjoint union of closed & irreducible sets of recurrent

general strategy



Finally,

Thm. Assume $S_R \neq \phi$, then

$$S_R = \bigcup_{i=1}^{k} C_{x_i} \qquad (k<\infty \text{ or } k=\infty)$$

disjoint union

closed & irreducible set of recurrent states.

Thm. 1°. If C is irreducible & closed set, then

either $C \subset S_R$ or $C \subset S_T$.

2°. If C is a finite, irreducible closed set, then

$$C \subset S_R.$$

Pf. : 1°. Contradiction

2°. Consequence of the fact

" A MC over the finite state space must contain at least one recurrent state"

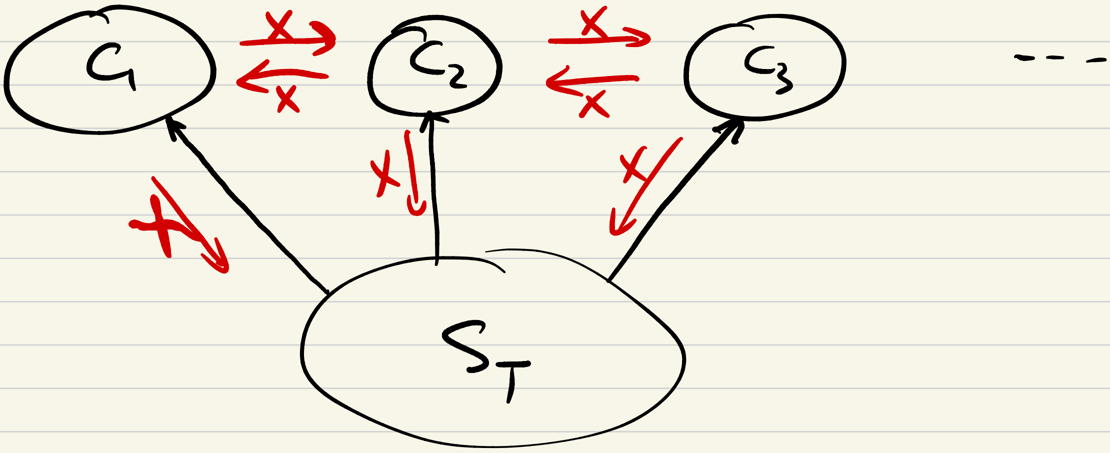
#

$$S = S_T \cup S_R$$

State space

$$= S_T \cup \Big(\underbrace{\bigcup_{i=1}^{k \leq \infty}}_{\text{disjoint}} \; C_i \Big)$$

C_i: closed & irreducible of recurrent states



Reordering states by the decomposition

for instance $S = S_T \cup C_1 \cup C_2$

(case $k = 2$)

$$\tilde{P} \overset{\text{def.}}{=} \begin{array}{c|ccc} & \boxed{C_1} & \boxed{C_2} & \boxed{S_T} \\ \hline \boxed{C_1} & & & \\ \boxed{C_2} & & & \\ \boxed{S_T} & & & \end{array} \quad \begin{array}{c} \\ \boxed{C_1} \\ \boxed{C_2} \\ \boxed{S_T} \end{array} \begin{array}{c} \begin{array}{ccc} \boxed{C_1} & \boxed{C_2} & \boxed{S_T} \end{array} \\ \left(\begin{array}{c|c|c} P_1 & 0 & 0 \\ \hline 0 & P_2 & 0 \\ \hline * & * & Q \end{array} \right) \end{array}$$

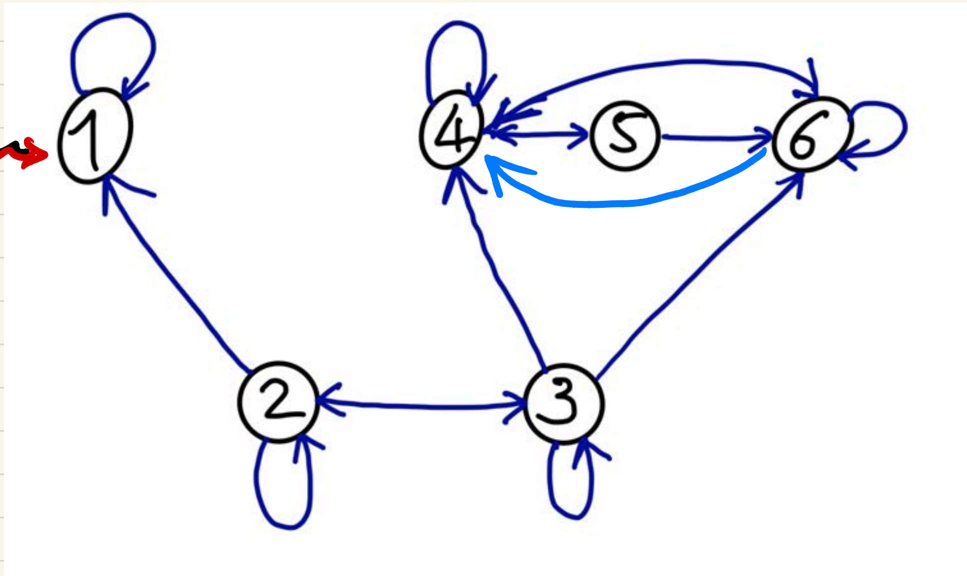
$\tilde{P}$: Markov matrix

Canonical form of the Markov matrix P.

An example: Find the canonical Markov matrix for

$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4}
\end{bmatrix}
\end{array}
$$

absorbing state



1°. {4.5.6} is closed (∵ no state in this set leads to a state outside it.)

$4 \xrightarrow{\text{leads to}} 5 \to 6 \to 4$; ∴ {4.5.6} is irreducible

∴ {4.5.6} is an irreducible closed set

∵ it is finite

$\therefore\ C_2 \overset{\text{def.}}{=} \{4, 5, 6\}$ is an irreducible closed set of recurrent state

$2^\circ.\quad C_1 = \{1\}$: irreducible closed set of recurrent states.

$3^\circ.\quad S_T = \{2, 3\}$

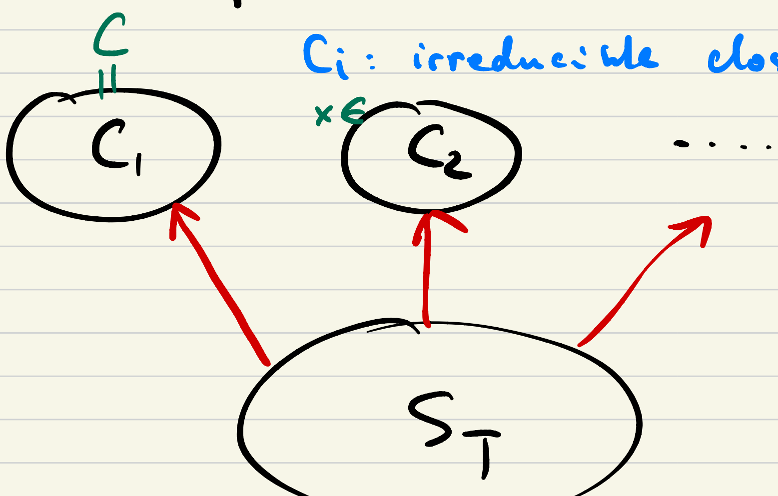
$$S = \underset{\substack{\| \\ \{2,3\} \\ \| \ \ \| \\ e \ \ f}}{S_T} \cup \underset{\substack{\| \\ \{1\} \\ \| \\ a}}{C_1} \cup \underset{\substack{\| \\ \{4,5,6\} \\ \| \ \| \ \| \\ b \ c \ d}}{C_2}$$

Find the Canonical form

$$\begin{array}{cc} & \begin{array}{c|ccc|cc} C_1 & & C_2 & & S_T & \\ \boxed{a=1} & \boxed{b=4} & \boxed{c=5} & \boxed{d=6} & \boxed{e=2} & \boxed{f=3} \end{array} \\ \begin{array}{cc} C_1 & \boxed{a=1} \\ \hline & \boxed{b=4} \\ C_2 & \boxed{c=5} \\ & \boxed{d=6} \\ \hline S_T & \boxed{e=2} \\ & \boxed{f=3} \end{array} & \left[\begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ \hline \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{array}\right]. \end{array}$$

the last computational issue :

C_i : irreducible closed set of recurrent sates



Q.: Compute for $C = \underset{\sim}{\text{some } C_i}$

irreducible closed set of recurrent states.

$$\rho_C(x) \stackrel{\text{def.}}{=} P_x\left(T_C < \infty\right), \quad x \in S$$

Recall: $T_C \stackrel{\text{def.}}{=} \min\{n \geq 1 : X_n \in C\}$

is the 1st positive time the chain hits C.

is the prob. that the chain from x enters into C in finite time

Observe:

$$\underbrace{\boxed{\rho_C(x)}}_{\text{different from } \rho_{xy} = P_x(T_y < \infty)}^{P_x(T_C<\infty)} = \begin{cases} 1 & \text{if } x \in C = C_i \\ 0 & \text{if } x \in S_R \text{ is still recurrent, but } x \notin C = C_i \\ ? & \text{if } x \in S_T \text{ is a transient.} \end{cases}$$

We need to determine:

$$\rho_C(x) = P_x(T_C < \infty), \quad x \in S_T.$$

C ↑ irreducible closed set of recurrent states

Claim: $\rho_C(x) = P_x(1 \leq T_C < \infty)$

$x \in S_T$

$$= P_x(T_C = 1) + P_x(2 \le T_C < \infty)$$

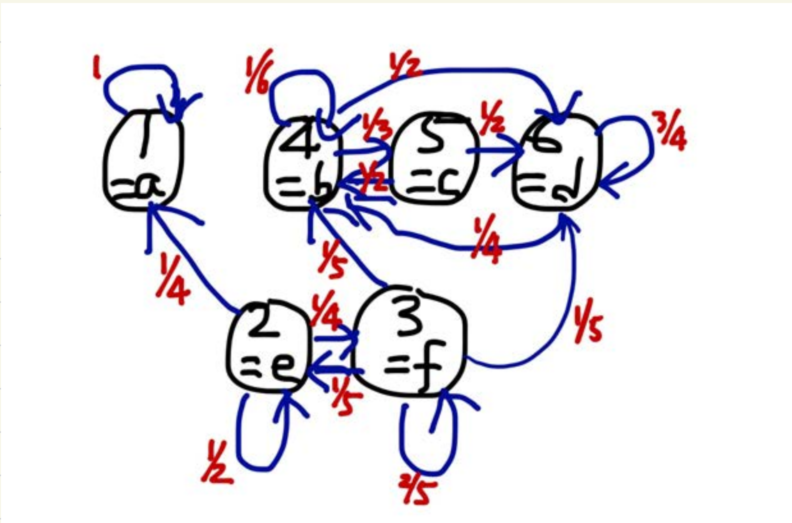
↑ use the two-steps strategy

$$= \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y) \underbrace{\rho_C(y)}_{= P_y(1 \le T_C < \infty)}$$

this is a linear system of equations for the unknowns $\{\rho_C(x)\}_{x \in S_T}$

Thm: If S_T (set of all transient states) is finite, then the linear system admits one and only one solution.

pf. omitted. #



$C_1 = \{1\}$

$C_2 = \{4, 5, 6\}$

$S_T = \{2, 3\}$

Compute $\rho_{C_2}(x)$, $x \in S_T$

i.e. look for

two unknowns
$$\begin{cases} x \overset{\text{def.}}{=} \rho_{C_2}(2) = P_2(T_{C_2} < \infty) \\ y \overset{\text{def.}}{=} \rho_{C_2}(3) = P_3(T_{C_2} < \infty) \end{cases}$$

$$\overset{\text{"}2\to C_2\text{"}}{x} = \underbrace{\overset{\text{"}2\to 4\text{"}}{0} + \overset{\text{"}2\to 5\text{"}}{0} + \overset{\text{"}2\to 6\text{"}}{0}}_{\text{one-step transition}} + \underbrace{\overset{\text{"}2\to 2\text{"} * \text{"}2\to C_2\text{"}}{\tfrac{1}{2}\cdot x} + \overset{\text{"}2\to 3\text{"} * \text{"}3\to C_2\text{"}}{\tfrac{1}{4}\cdot y}}_{\text{two-steps transition}}$$

Similiarly,

$$\overset{\text{"}3\to C_2\text{"}}{y} = \left(\overset{\text{"}3\to 4\text{"}}{\tfrac{1}{5}} + \overset{\text{"}3\to 5\text{"}}{0} + \overset{\text{"}3\to 6\text{"}}{\tfrac{1}{5}}\right) + \left(\overset{\text{"}3\to 2\text{"}\cdot\text{"}2\to C_2\text{"}}{\tfrac{1}{5}\cdot x} + \overset{\text{"}3\to 3\text{"}\cdot\text{"}3\to C_2\text{"}}{\tfrac{2}{5}\cdot y}\right)$$

$$\therefore \begin{cases} x = \frac{1}{2}x + \frac{1}{4}y \\ y = \frac{2}{5} + \left(\frac{1}{5}x + \frac{2}{5}y\right) \end{cases}$$

$$\Rightarrow \cdots \Rightarrow x = \frac{2}{5},\quad y = \frac{4}{5} \quad \#$$

Remarks:

(i) If S_T is finite, then for any $x \in S_T$,

$$\underbrace{\sum_{i=1}^{k} \rho_{C_i}(x) = 1}$$

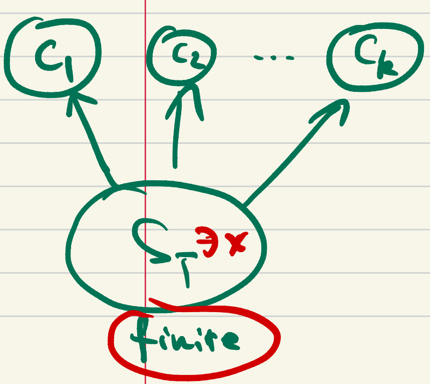
where $S_R = \bigcup_{i=1}^{k} C_i$, $k < \infty$ or $= \infty$

disjoint

irred + closed of recurrent

$\Updownarrow$

The chain from x hits some C_i in finite time "for sure"



(i.e. with prob $=1$)

Heuristic argument:

The total times the chain from x visit any transient state have to be finite for sure, and after then the chain must hit some C_i.

$$LHS = \sum_{i=1}^{k} \rho_{C_i}(x)$$

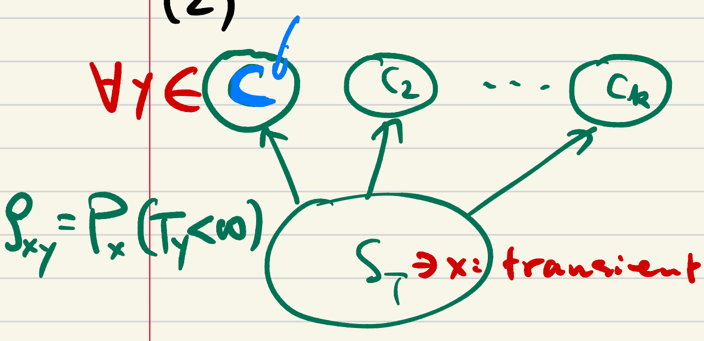
$$= \sum_{i=1}^{k} P_x\left(1 \le T_{C_i} < \infty\right)$$

$$= \cdots = P_x\left(T_{S_R} < \infty\right) = 1 \quad \#$$

We only have finite transient states. #

(2)

Inside C, $\rho_{ab}=1$, $\forall a,b \in C$



Recall:

Recurrent $x \xrightarrow{\rho_{xy}=1} y$ recurrent, $\rho_{yx}=1$

: $\rho_C(x) = P_x\left(T_C < \infty\right)$

Claim: $\rho_C(x) = \rho_{xy}, \quad \forall y \in C.$

Pf. $\{1 \le T_C < \infty\} \supseteq\subseteq \{1 \le T_y < \infty\}$. #