Announcement on Feb. 3rd :

① Feb. 8th (Mon.)

* Course lecture : 12:30pm – 2:15pm
  The same zoom information

* Tutorial will be cancelled on this day

② Feb. 20th (Sat.) :

* One added course lecture : 9:30am – 12:00pm

③ Feb. 22nd (Mon)

* Course lecture will be cancelled, but
* Tutorial is on schedule

Continue : 3rd Computational issue :

$N(y) \overset{\text{def.}}{=}$ the total no of times that the chain X_n $(n \geq 1)$ visits y.

r.v.

i.e. at the positive time

$*$ $N(y)_\omega \in \{0, 1, 2, \cdots\} \cup \{\infty\}$

$\{N(y) = 0\} = \{$ chain X_n $(n \geq 1)$ never visits y $\}$

$\{N(y) = k\} = \{$ chain X_n $(n \geq 1)$ visits y exactly for k times $\}$

$k = 1, 2, 3, \cdots$

$\{N(y) = \infty\} = \{$ chain X_n $(n \geq 1)$ visits y for infinite times $\}$

$$* \quad N(y) = \sum_{n=1}^{\infty} 1_y(X_n),$$

$$1_y(x) \overset{\text{def.}}{=} \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

indicator function

Prop.

① $P_x(\infty \geq N(y) \geq 1) = P_x(T_y < \infty) \overset{\checkmark}{=} \rho_{xy}$

Pf. : $\{N(y) \geq 1\} = \{T_y < \infty\}$

X_n $(n \geq 1)$ visits y at least once

X_n visits y at some finite positive time

$$P_x(N(y)=0) = 1 - \rho_{xy}$$

pf. : $\Omega = \{0 \leq N(y) < \infty\} \cup \{N(y)=\infty\}$

$$\{N(y)=0\} = \{1 \leq N(y) \leq \infty\}^c$$

$$P_x(N(y)=0) = 1 - P_x(1 \leq N(y) \leq \infty)$$
$$= 1 - \rho_{xy}. \quad \#$$

② For any $m \geq 1$, $P_x(N(y) \geq m) = \rho_{xy}\,\rho_{yy}^{m-1}$

$$P_x(N(y)=m) = \rho_{xy}\,\rho_{yy}^{m-1}(1-\rho_{yy})$$

pf. $m=1$: ok

$m=2$: to show $P_x(N(y) \geq 2) = \rho_{xy}\,\rho_{yy}$

$$\{N(y) \geq 2\} = \bigcup_{k \geq 1} \bigcup_{n \geq 1} \{\text{the chain from } x \text{ visit } y \text{ for the 1st time at time } k\text{, and after then, the chain visits } y \text{ again for the 2nd time after } n \text{ units of time.}\}$$

Chain X_n $(n \geq 1)$ visit y at least twice

(disjoint union)

$\Downarrow$

$$P_x(N(y) \geq 2) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \boxed{P_x(\cdots)}$$

$P(A \cap B) = \underbrace{P(A)}\, \underbrace{P(B \mid A)}$

|| claim by the Markov property

$$P_x(T_y = k)\, \underbrace{P_y(T_y = n)}$$

$$= \left[\sum_{k=1}^{\infty} P_x(T_y = k) \right] \cdot \left[\sum_{n=1}^{\infty} P_y(T_y = n) \right]$$

$$= P_x(1 \le T_y < \infty)\; P_y(1 \le T_y < \infty)$$

$$= \rho_{xy}\, \rho_{yy}. \qquad \#$$

$m \geq 3$: Use the same idea (Exercise!)

For the 2nd identity, notice

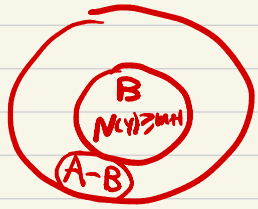
$$\{N(y) = m\} = \underbrace{\{N(y) \geq m\}}_{A} - \underbrace{\{N(y) \geq m+1\}}_{B}$$

Here : $\underbrace{B \subset A}$

$$\therefore \; P(N(y) = m)$$

$$= P(A) - P(B)$$

$$\overset{\text{1st identity}}{=} \rho_{xy}\, \rho_{yy}^{m-1} - \rho_{xy}\, \rho_{yy}^{(m+1)-1}$$



A : $N(y) \geq m$

$$= \underbrace{\rho_{xy}\, \rho_{yy}^{m-1}}\; \underbrace{(1 - \rho_{yy})}_{\substack{\| \\ P_y(T_y = \infty)}}. \qquad \#$$

$(A-B) \cup B = A$

$\uparrow$ disjoint

$\underbrace{P(A-B) = P(A) - P(B)}$

Prop. $E_x(N(y)) = \sum_{n=1}^{\infty} P^n(x, y)$.

$\underbrace{E_x}$: $\underbrace{N(y)}$ r.v. : no of times X_n $(n \geq 1)$ visits y

expected no of times chain from x visits y at positive times.

Note : It could occur that

$E_x(N(y)) = \infty$

pf. $E_x(N(y))$

$= E_x\left(\sum_{n=1}^{\infty} 1_y(X_n)\right)$

$= \sum_{n=1}^{\infty} E_x(1_y(X_n))$ $\qquad 1_y(X_n) = \begin{cases} 1 & X_n = y \\ 0 & X_n \neq y \end{cases}$

$= \sum_{n=1}^{\infty} \left[1 \cdot P_x(X_n = y) + \cancel{0 \cdot P_x(X_n \neq y)}\right]$

$= \sum_{n=1}^{\infty} P_x(X_n = y)$ $\qquad \sum_i \sum_j \underbrace{a_{ij}}_{\geq 0} = \sum_j \sum_i a_{ij}$

$= \sum_{n=1}^{\infty} P^n(x, y)$. #

The following goal :

Use the previous to characterize whether a state y is recurrent / transient

$\rho_{yy} = 1 \qquad \rho_{yy} < 1$.

Theorem. (i) y is transient iff

$$P_y(N(y)=\infty)=0$$

In case y is transient,

$$E_x(N(y))=\frac{\rho_{xy}}{1-\rho_{yy}}<\infty, \quad \forall x\in S.$$

Pf. Note:

$$(*)\quad P_x(N(y)=\infty) = \lim_{m\to\infty} \underbrace{P_x(N(y)\geq m)}$$

$$= \lim_{m\to\infty} \rho_{xy}\boxed{\rho_{yy}^{m-1}}$$

$$= \begin{cases} 0 & \text{if } \underline{\rho_{yy}<1} \\ & \quad y \text{ transient} \\ \rho_{xy} & \text{if } \rho_{yy}=1 \\ & \quad (y \text{ recurrent}) \end{cases}$$

(i) y is transient

$$\overset{\text{def.}}{\iff} \rho_{yy}<1$$

$$\overset{(*)}{\iff} P_y(N(y)=\infty)=0.$$

$\Rightarrow$ ok

$\Leftarrow$ ok

Moreore, if y is transient, then

$$P_x(N(y)=\infty) \overset{(*)}{=} 0$$

Now

$$E_x(N(y)) = \sum_{m=0}^{\infty} m\, P_x(N(y)=m)$$

$$= \sum_{m=1}^{\infty} m \, P_x (N(y) = m)$$

$$= \sum_{m=1}^{\infty} m \, \rho_{xy} \, \rho_{yy}^{m-1} \, (1 - \rho_{yy})$$

Recall : (Exercise)

$$\sum_{m=0}^{\infty} t^m = \frac{1}{1-t} \quad (|t|<1)$$

$$\sum_{m=1}^{\infty} m t^{m-1} = \left(\frac{1}{1-t}\right)' = \frac{1}{(1-t)^2}$$

$$= \rho_{xy} (1-\rho_{yy}) \sum_{m=1}^{\infty} m \, \rho_{yy}^{m-1} \quad (0 \le \rho_{yy} < 1)$$

y transient

$$= \rho_{xy} (1-\rho_{yy}) \frac{1}{(1-\rho_{yy})^2}$$

$$= \frac{\rho_{xy}}{1-\rho_{yy}} < \infty$$

$(\rho_{yy} < 1)$

(ii) y is recurrent iff $P_y (N(y) = \infty) = 1$,

iff $E_y (N(y)) = \infty$.

pf: y recurrent

$\overset{\text{def.}}{\Longleftrightarrow} \rho_{yy} = 1$

$\overset{(*)}{\Longleftrightarrow} P_y (N(y) = \infty) = 1$ $\left\{ \begin{array}{l} \Rightarrow \text{ ok} \\ \Leftarrow \text{ ok} \end{array} \right.$

$\overset{(**)}{\Longleftrightarrow} E_y (N(y)) = \infty$

pf of $(**)$: "$\Rightarrow$" obvious.

$$E_y(N(y)) \ge 1 \cdot \underbrace{P_y (N(y) = \infty)}_{= \infty}$$

$\Leftarrow$ to show:

y is recurrent

otherwise, y transient

then $E_y(N(y)) = \frac{\rho_{yy}}{1-\rho_{yy}} < \infty$

contradiction! #

Remark: If y is recurrent, then $\forall x \in S$,

$$E_x(N(y)) = \begin{cases} 0 & \text{if } \rho_{xy} = 0 \\ \infty & \text{if } \rho_{xy} > 0 \end{cases}$$

#

Corollary:

If S is finite, then the chain must have at least one recurrent state.

Pf. : otherwise, all states are transient.
(finite)

Then, for any x, y that are transient,

$$\sum_{n=1}^{\infty} P^n(x,y) = E_x(N(y)) \overset{y \text{ transient}}{=} \frac{\rho_{xy}}{1-\rho_{yy}} < \infty$$

$$\therefore \lim_{n\to\infty} P^n(x,y) = 0, \quad \forall x, y \in S \text{ (finite)}$$

$$0 = \underbrace{\sum_{y \in S}}_{\text{finite sum}} \lim_{n\to\infty} P^n(x, y)$$

$$= \lim_{n\to\infty} \sum_{y \in S} P^n(x, y)$$

$$= \lim_{n\to\infty} 1 \qquad (\because \; P^n : \text{Markov matrix})$$

$$= 1, \qquad \text{Contradiction!} \qquad \#$$