Chap 1: Markov Chain

— a discrete-in-time stochastic process with the Markovian property

$$X_0, X_1, X_2, \cdots, X_n, \cdots$$

or

$$\{X_k\}_{k=0}^{\infty}$$

Suppose :

① take values in the same state space

$$S = \{0, 1, 2, \cdots, N\}$$

finite or ∞

(countable)

each integer denote a state of the system

② defined on a common prob. space

$$(\Omega, \mathcal{F}, P)$$

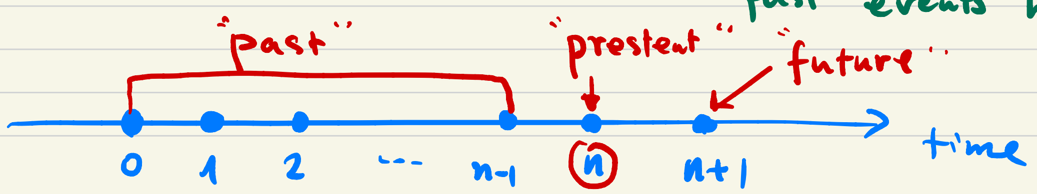
Markovian property :

$$P(\underbrace{X_{n+1} = x_{n+1}}_{\text{future}} \mid \underbrace{X_0 = x_0, \cdots, X_{n-1} = x_{n-1}}_{\text{past}}, \underbrace{X_n = x_n}_{\text{present}})$$

$$= P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

$x_0, x_1, \cdots, x_n, x_{n+1} \in S$
states

↳ Roughly says : Given the "present" event, the "past" events have no influence on the "future" event.



note: $P(x,y) \overset{\text{def.}}{=} P(X_{n+1}=y \mid X_n = x)$

$\forall x, y \in S$

$\longrightarrow$ transition function

We always assume:

$\underbrace{P(x,y)}_{\text{transition function}}$ independent of n,

i.e. the chain is time-homogeneous.

$P(x,y)$, $x, y \in S = \{0, 1, 2, \dots, N\}$ $\downarrow$ finite or ∞

① $P(x,y) \geq 0$;

② $\sum_{y \in S} P(x,y) = 1$.

pf: ① $P(x,y) = P(X_{n+1}=y \mid X_n = x) \geq 0$.

② $\sum_{y \in S} P(X_{n+1}=y \mid X_n = x)$

$\{X_n = x\} = \bigcup_{y \in S} \{X_n = x, X_{n+1} = y\}$

$\uparrow$ disjoint union

$= P(X_{n+1} \in \Omega \mid X_n = x)$

$= 1$

$\bigcup_{y \in S} \{X_{n+1} = y\} = \Omega$

Notation:

Each Row sum $= 1$

$$P \overset{\text{def.}}{=} [P(x,y)]_{x,y \in S} = \begin{array}{c|cccc} & 0 & 1 & \cdots & N \\ \hline 0 & P(0,0) & P(0,1) & \cdots & P(0,N) \\ 1 & P(1,0) & P(1,1) & \cdots & P(1,N) \\ 2 & P(2,0) & P(2,1) & \cdots & P(2,N) \\ \vdots & \vdots & \vdots & & \vdots \\ N & P(N,0) & P(N,1) & \cdots & P(N,N) \end{array}$$

Markov transition matrix

Jan 18: Recall:

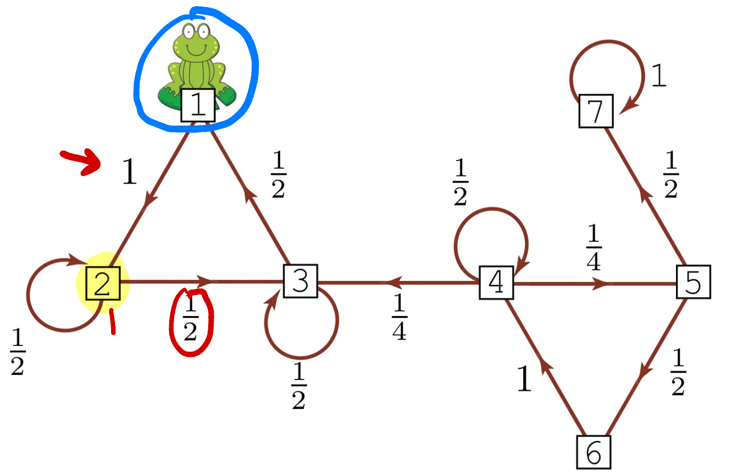
Markov matrix : $P = [P(x,y)]_{x,y \in S}$

- $P(x,y) = P(X_{n+1} = y \mid X_n = x) \geq 0$;
- Row sum $= \sum_{y \in S} P(x,y) = 1, \ \forall x \in S$.

i.e. the sum of probabilities of transitions from x to all states $= 1$

$S = \{1, 2, 3, 4, 5, 6, 7\}$

$P(X_{n+1} = 2 \mid X_n = 1) = 1$



$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

examples:

- Setup the model as Markov chain
- Find the Markov matrix

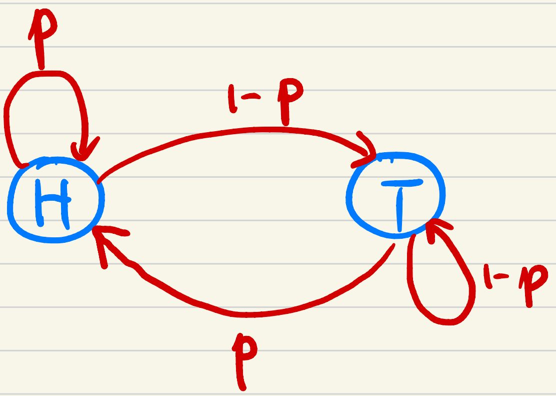
e.g. 1: Toss a biased coin:

H T

$\{X_n\}_{n=0}^{\infty}$: i.i.d. (independent identically distributed)

$$X_n = \begin{cases} H & \text{with } p \\ T & \text{with } 1-p \end{cases}$$

$n = 0, 1, \ldots$



$$\begin{array}{c|cc} & H & T \\ \hline H & & \\ T & & \end{array} \quad \begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$

transition matrix

e.g. 2 Two-state Markov chain

Check the state of a machine,

$$X_n = \begin{cases} 0 & \text{if "broken"} \\ 1 & \text{if "in operation"} \end{cases}$$

$n = 0, 1, 2, \ldots$

$$p = P(X_{n+1} = 1 \mid X_n = 0)$$

$0 \le p \le 1$

is the prob. that
given that it is "broken",
"it will be in operation" in the next day.

$$q = P(X_{n+1} = 0 \mid X_n = 1)$$

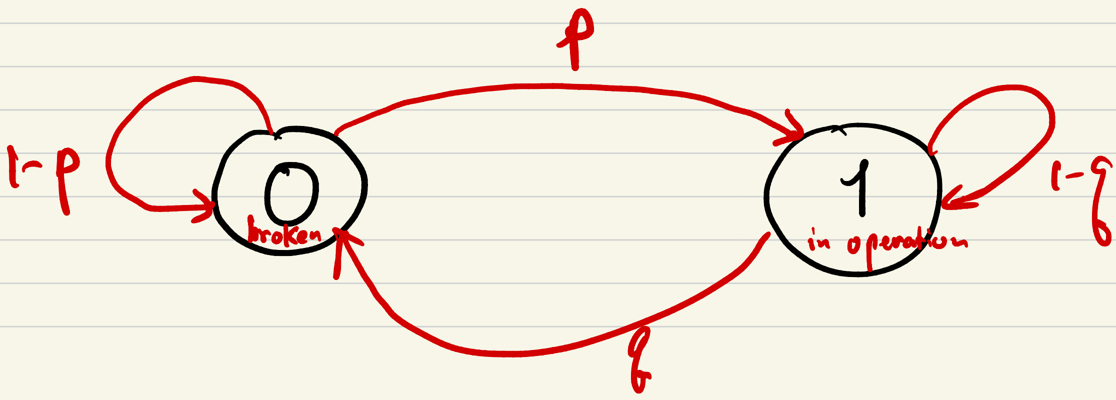
$0 \le q \le 1$

the prob. that
given that it is "in operation",
it will be "broken" on the next day.

Markov matrix

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline \text{"broken"} \leftarrow 0 & 1-p & p \\ \text{"in operation"} \leftarrow 1 & q & 1-q \end{array}_{2\times 2}$$

Row sum $\equiv 1$



general question:

We know: $\{X_n\}_{n=0}^{\infty}$, $S=\{0,1\}$

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-p \end{pmatrix} \end{array}$$

$0 \leq p, q \leq 1$.

Can you compute

$$\underset{\text{"broken"}}{P(X_n = 0)} \quad \text{or} \quad \underset{\text{"in operation"}}{P(X_n = 1)}$$

For instance,

$$\Omega = \{X_{n-1}=0\} \cup \{X_{n-1}=1\}$$

$$P(X_n=0) = P(X_n=0,\ X_{n-1}=0 \text{ or } 1)$$

$$= \underbrace{P(X_n=0 \mid X_{n-1}=0)}_{=1-p} P(X_{n-1}=0)$$

$$+ \underbrace{P(X_n=0 \mid X_{n-1}=1)}_{=q} \underbrace{P(X_{n-1}=1)}_{=1-P(X_{n-1}=0)}$$

$$P(X_n=0)+P(X_n=1) = P(\Omega)=1$$

$$= \underbrace{(1-p-q)}_{\overset{\text{def.}}{=} a \in (-1,1)} P(X_{n-1}=0) + q$$

Assume:

$0<p, q<1$

$0<p+q<2$

$-1 < a = 1-(p+q) < 1$

$\Downarrow$

$|a| < 1$

Sum:

$$P(X_n=0) = a\,P(X_{n-1}=0) + q$$

$a = 1-p-q \in (-1, 1)$

induction in n

$= \cdots$

$$= a^n P(X_0=0) + \underbrace{(a^{n-1}+\cdots+a+1)}_{=\frac{1-a^n}{1-a}} q$$

$$\therefore \quad P(X_n=0) = a^n P(X_0=0) + \frac{1-a^n}{1-a} q$$

$a = 1-(p+q)$.

Similarly (or $P(X_n=1) = 1 - P(X_n=0)$)

$$P(X_n=1) = a^n P(X_0=1) + \frac{1-a^n}{1-a} p$$

$$\text{“}+\text{”} = a^n \cdot 1 + \frac{1-a^n}{1-a} \cancel{(p+q)}$$

$1-a = 1-[1-(p+q)] = \cancel{p+q}$

$= a^n + (1-a^n)$

$= 1.$

Remark: Recall $|a|<1$ $\Rightarrow a^n \xrightarrow[n\to\infty]{} 0$ $\left(\because a = 1-(p+q)\right)$

$$\lim_{n\to\infty} P(X_n=0) = \frac{q}{p+q}$$

the prob. in the long-run ($n \to \infty$)

that the machine is "broken"

$$\lim_{n \to \infty} P(X_n = 1) = \frac{p}{p+q}$$

$$P = \begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} 1-p & p \\ q & 1-p \end{bmatrix}$$

i.e.

$$\lim_{n \to \infty} \underbrace{\left[P(X_n = 0), \; P(X_n = 1) \right]}_{\text{pdf of } X_n} = \underbrace{\left[\frac{q}{p+q} , \; \frac{p}{p+q} \right]}_{\text{Limit distributi}}$$

(1×2 prob. row vector)