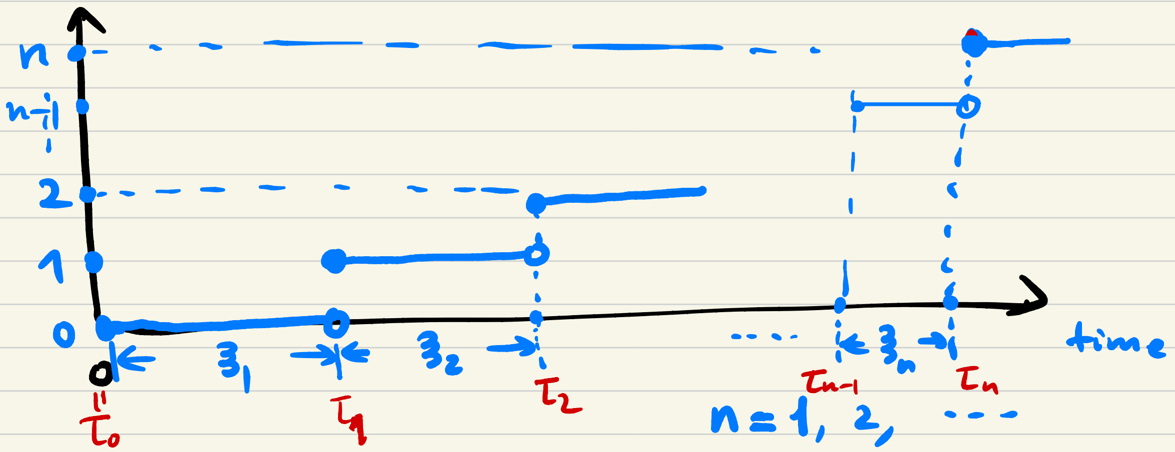
§2. Poisson process

(a nontrivial example for a time-homoge. MJP)



Model: Counting arrivals

$\xi_n \overset{\text{i.i.d.}}{\sim} \xi$: waiting time for one arrival

$$P(\xi > t) = e^{-\lambda t}, \quad \forall\, t \geq 0,$$

$$\lambda = \frac{1}{E(\xi)} > 0$$

$$\tau_0 \overset{\text{Convention}}{=} 0$$

$$\tau_n \overset{\text{def.}}{=} \xi_1 + \xi_2 + \cdots + \xi_n, \quad n = 1, 2, \cdots$$

↳ the waiting time for the n^{th} arrival.

Set $X_0 = 0$,

$X_t \overset{\text{def.}}{=}$ total no of arrivals up to time $t > 0$

$$= \max\{ n \geq 0 : \tau_n \leq t \},$$

$\forall\, t > 0$

We see: $\{X_t\}_{t\geq 0}$ is a JP

to show:

1°. X_t is a Poisson r.v. with parameter $=\lambda t$

2°. Markov property is satisfied.

$\rightsquigarrow$ then we call it Poisson Process.

#

March 27:

Thm: X_t is a r.v. with Poisson distribution with parameter λt, i.e.

$$P(X_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n=0,1,\cdots$$

$t > 0$.

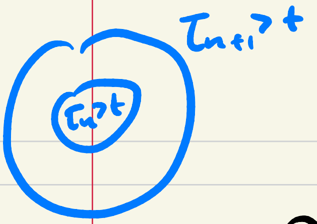
Proof. Observe

$$\{X_t = n\} = \{T_n \leq t < T_{n+1}\}$$

$\uparrow$ n arrivals up to t

* up to t, get the n^{th} arrival

* the time for the $(n+1)^{th}$ arrival must be after t

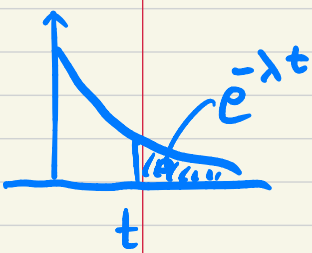


$$= \{\tau_{n+1} > t\} - \{\tau_n > t\}$$

$$\therefore \quad P(X_t = n) = P(\tau_{n+1} > t) - P(\tau_n > t) \quad (*)$$

Check $n=0$: $\boxed{t>0}$

$$P(X_t = 0) = \underbrace{P(\tau_1 > t)}_{\substack{= P(\xi > t) \\ = e^{-\lambda t}}} - \underbrace{P(\tau_0 > t)}_{=0}, \quad \tau_0 = 0$$



$$= e^{-\lambda t}, \quad \forall t > 0$$

Claim (Induction in $n \geq 1$) :

$$P(\tau_n > t) = e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}, \quad \forall t > 0.$$

if you agree, plug it to (*) . #

pf of claim :

$n=1$: $P(\tau_1 > t) = P(\xi_1 > t) = e^{-\lambda t}$

Assume "TRUE" for $n \geq 1$,

to show "TRUE" for $n+1$:

$$P(\tau_{n+1} > t)$$

$$= P(\tau_n + \xi_{n+1} > t)$$

$$= P(\xi_{n+1} > t) + P(\underset{= s}{\xi_{n+1}} \leq t, \ \tau_n + \overset{s =}{\xi_{n+1}} > t)$$

$\tau_n > t-s$ $\Uparrow$

$P(\xi_{n+1} = s) = \lambda e^{-\lambda s}$

$$= e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \underbrace{P(T_n > t-s)}_{\overset{\text{I.A.}}{=} e^{-\lambda(t-s)} \sum_{k=0}^{n-1} \frac{[\lambda(t-s)]^k}{k!}} \, ds$$

$$= e^{-\lambda t} + \underset{\text{Exercise}}{(\cdots)}$$

$$= e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} . \quad (\text{desired}) \qquad \#$$

Rk :

$E(X_t) = \lambda t$: average no of arrivals in $[0, t]$

$\dfrac{\lambda t}{t} = \lambda$: arrival rate

no of arriving persons/unit time

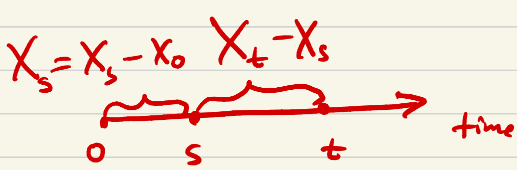
Prop. : The (Poisson) process

$$\{X_t\}_{t \geq 0}$$

defined above in terms of waiting time, Satisfies the following properties :

(i) $X_0 = 0$; (ok by def.)

(ii) For $0 < s < t$,



$$X_t - X_s$$

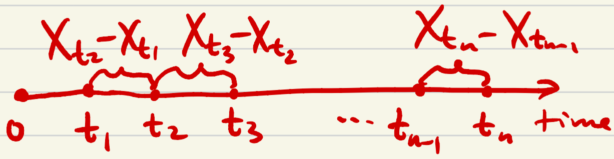
is a r.v. of Poisson distribution with mean $= \lambda(t-s)$, and

$X_t - X_s$ is independent of X_s

(iii) For $0 < t_1 \le \cdots \le t_n$,

$$X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2},\ \cdots,\ X_{t_n} - X_{t_{n-1}}$$

are independent.



Moreover,

$\{X_t\}_{t \ge 0}$ satisfies the Markov property.

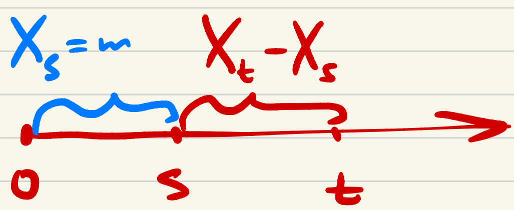
(so, it is MJP, called the Poisson process). #

Remark:

(i) (ii) (iii) are often used to directly give the definition of P.P. #

Proof (Sketch).

(ii) take $0 < s < t$



$$\bigcup_{m=0}^{\infty} \{X_s = m\} = \Omega$$

$$P(X_t - X_s = n)$$

$$= \sum_{m=0}^{\infty} P(\underbrace{X_t - X_s = n}_{\Updownarrow\ X_t = m+n}, \underbrace{X_s = m})$$

$$P(A \cap B) = P(A|B)P(B)$$

$$= \sum_{m=0}^{\infty} P(X_t = m+n \mid X_s = m)\, P(X_s = m)$$

$$= P(X_{t-s} = n \mid X_0 = 0)$$
$$= P_0(X_{t-s} = n)$$
$$= \sum_{m=0}^{\infty} P_0(X_{t-s} = n)\, P(X_s = m)$$
$$= P_0(X_{t-s} = n) \cdot 1 \,. \quad \#$$
$$= e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^n}{n!} \; : \; \text{Poisson}$$

Warning: $\underbrace{X_t - X_s}_{\text{no of arrivals in the period } [s,t]} \neq \underbrace{X_{t-s}}_{\text{no of arrivals in } [0,\, t-s]}$

To show: indep. of $X_t - X_s$ & X_s

$$P(X_t - X_s = n,\, X_s = m) \overset{?}{=} P(X_t - X_s = n) P(X_s = m)$$

Indeed,

$$\text{LHS} = P(X_t - X_s = n \mid X_s = m)\, P(X_s = m)$$

to show:

$$P(X_t - \underbrace{X_s}_{= m} = n \mid X_s = m) = P(X_t - X_s = n)$$

Indeed,

$$\text{LHS} = P(X_t = m+n \mid X_s = m)$$

$$= \cdots = P(X_{t-s} = n \mid X_0 = 0)$$

$$= P(X_{t-s} = n). \quad \#$$

Omit the proof of rest parts

((iii) and Markov)

(Tutorial). #