March 17:

§4. Periodicity

Def. Let $x \in S$,

$$d_x \overset{\text{def.}}{=} \text{g.c.d.}\{n \geq 1 : P^n(x,x) > 0\}$$

greatest common divisor

called the period of x.

note:

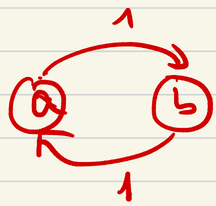
1°. $1 \leq d_x \leq \min\{n \geq 1 : P^n(x,x) > 0\}$.

2°. If $P(x,x) > 0$, then

$$1 \in \{n \geq 1 : P^n(x,x) > 0\}$$

$$\therefore \quad d_x = 1.$$

e.g. $P = \begin{array}{cc} a & b \\ \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \end{array}$



$P^{2n} = \begin{array}{c|c} & a \\ \hline a & \left(\begin{array}{cc} \underset{>0}{1} & 0 \\ 0 & 1 \end{array}\right) \end{array}$

$P^{2n+1} = \begin{array}{c|c} & a \\ \hline a & \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \end{array}$

$x = a$:

$$\text{g.c.d.}\{m \geq 1 : P^m(a,a) > 0\}$$

$$= \text{g.c.d.} \{ 2n : n = 1, 2, \cdots \}$$

$$= 2 .$$ period of state a.

Similarly,

$x = b$,

$$\text{g.c.d.} \{ m \geq 1 : P^m(b, b) > 0 \}$$
$$= \cdots = 2. \quad \#$$

Prop. For an irreducible MC, all states must have the same period.

Def. this period is called the period of the chain

Proof: $x \neq y$, to show : $d_x = d_y$

$$\begin{tikzcd} x \arrow[r, bend left, "\rho_{xy} > 0"] & y \arrow[l, bend left, "\rho_{yx} > 0"] \end{tikzcd}$$

$\Rightarrow \exists n_1 \geq 1$, s.t. $P^{n_1}(x, y) > 0$

$\Rightarrow \exists n_2 \geq 1$, s.t. $P^{n_2}(y, x) > 0$

then

$$P^{n_1+n_2}(x, x) = (P^{n_1} \cdot P^{n_2})(x, x)$$
$$\geq \underbrace{P^{n_1}(x, y)}_{>0} \underbrace{P^{n_2}(y, x)}_{>0}$$

$$\underset{\substack{\| \\ d_x}}{\text{g.c.d.}} \{ \underline{m \geq 1 : P^m(x, x) > 0} \} \quad \ni n_1 + n_2$$

$$\therefore d_x \mid n_1 + n_2$$

to show : $d_x \mid d_y = \text{g.c.d.} \{ m \geq 1 : P^m(y,y) > 0 \}$

take an arbitrary $n \geq 1$ s.t.

$$P^n(y,y) > 0$$

it sufficies to show : $d_x \mid n$

notice :

$$P^{n_1+n+n_2}(x,x) \geq \underbrace{\underbrace{P^{n_1}(x,y)}_{>0} \underbrace{P^n(y,y)}_{>0} \underbrace{P^{n_2}(y,x)}_{>0}}_{>0}$$

$\therefore \quad d_x \mid n_1 + n + n_2$

$\therefore \quad d_x \mid (n_1 + n + n_2) - (n_1 + n_2) = n$

Conversely, using the same,

$$d_y \mid d_x .$$

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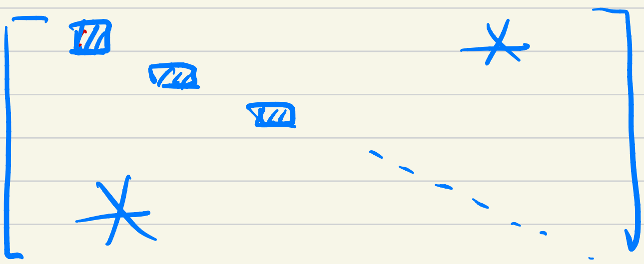
Remark. : Irreducible MC :

If $d = 1$, then this irreducible MC is said to be aperiodic.

e.g.

$\underset{\text{irreducible}}{P} =$ 

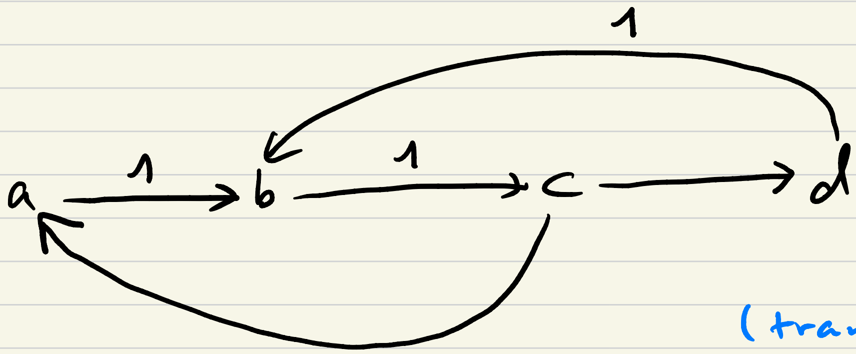
diagonal entries

If some diagonal entry > 0

then $d = 1$ $\therefore$ chain aperiodic. #

Example #1. :

$$\text{Markov } P = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \\ 0 & * & 0 & 0 \end{bmatrix} \qquad * > 0$$



(transition graph)

- irreducible MC.

- $d = d_a = ? = \text{g.c.d.}\{m \geq 1 : P^m(x, x) > 0\}$

$$= \text{g.c.d.}\{\underbrace{3n : n = 1, 2, \cdots}_{\text{By observation}}\}$$

$$= 3.$$

$$P^m(x,x) = \begin{cases} > 0 & m = 3n \quad \forall n = 1, 2, \cdots \\ = 0 & \text{otherwise} \end{cases}$$

Alternatively, you can compute :

$$P^{3n+1} = \begin{array}{cc} & \begin{array}{cccc} a & & & \end{array} \\ \begin{array}{c} a \\ \\ \\ \\ \end{array} & \begin{bmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \\ 0 & * & 0 & 0 \end{bmatrix} \end{array}$$

$$P^{3n+2} = \begin{array}{cc} & \begin{array}{cccc} a & & & \end{array} \\ \begin{array}{c} a \\ \\ \\ \\ \end{array} & \begin{bmatrix} 0 & 0 & * & 0 \\ * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \end{bmatrix} \end{array}$$

$$P^{3n} = \begin{array}{cc} & \begin{array}{cccc} a & & & \end{array} \\ \begin{array}{c} a \\ \\ \\ \\ \end{array} & \begin{bmatrix} * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \end{bmatrix} \end{array}, \qquad n = 1, 2, \cdots$$

$$\{m \geq 1 : P^m(a,a)\}$$

$$= \{3n : n = 1, 2, \cdots\}.$$

$$\text{g.c.d. } \{\ldots\} = 3 = d. \qquad \#$$

Example # 2.

BD MC : irreducible

$$P = \begin{array}{cc} \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{bmatrix} r_0 & p_0 & & & \\ q_1 & r_1 & p_1 & & 0 \\ & q_2 & r_2 & p_2 & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \end{array}$$

all $p_x > 0$, all $q_x > 0$

Case 1 : If some $r_x > 0$,

(some diagonal entry of $P > 0$)

Then chain aperiodic

Case 2 : otherwise, all $r_x \equiv 0$.

$$\begin{tikzcd} x-1 & x \arrow[l, "q_x"'] \arrow[r, "p_x"] & x+1 \end{tikzcd}$$

$p_x + q_x = 1$.

$\therefore$ chain from x return to x

ONLY in $2m$-steps

(even number of)

$$\underbrace{\{ n \geq 1 : P^n(x,x) > 0 \}}_{\substack{\Downarrow ? \\ 2}} \underset{\text{subset}}{\subset} \{ 2m : m = 1, 2, \cdots \}$$

Look at $x=0$:

$$\begin{tikzcd} 0 \arrow[r, shift left, "p_0 > 0"] & 1 \arrow[l, shift left, "q_1 > 0"] \arrow[r, "p_1"] & 2 \end{tikzcd}$$

$$P^2(0,0) = \sum_{x \in S} P(0,x) P(x,0)$$

$$= P(0,1) P(1,0)$$

$$= \underbrace{p_0}_{>0} \underbrace{q_1}_{>0} > 0$$

$$2 \in \{ n \geq 1 : P^n(0,0) > 0 \} \subset \{ \text{even} \}$$

$$\text{g.c.d.} \{ n \geq 1 : P^n(0,0) > 0 \} = 2 = d$$

period.

Continue on March 22:

Thm. (disscuss the long-run behavior $P^n(x,y)$ as $n \to \infty$)

Consider an irreducible (+)-recurrent MC $\{X_n\}_{n\geq 0}$ with state space S and transition function $P(x,y)$. Let π be the unique SD of this chain. Then,

(i) If chain is aperiodic, then

$$\lim_{n\to\infty} P^n(x,y) = \pi(y), \qquad \forall\, x,\ y \in S.$$

(ii) If chain has period $d \geq 2$, then

$\forall\, x, y \in S$, $\exists\, r = r(x,y) \in \{0, 1, \dots, d-1\}$ s.t.

$$P^n(x,y) = \begin{cases} 0 & \text{if } n \bmod d \neq r \\ \xrightarrow[n\to\infty]{} d\pi(y) & \text{if } n \bmod d = r \end{cases}$$

Remark: In case (ii): Once x, y are given, we have such

$$0 \leq \underbrace{r = r(x,y)}_{\text{integer}} \leq d-1$$

Such that

$$\left.\begin{array}{l} P^{md}(x,y) \\ P^{md+1}(x,y) \\ \vdots \\ P^{md+(d-1)}(x,y) \end{array}\right\} \text{ all } \equiv 0 \text{ except for only } \underbrace{\text{one term}}_{(n \bmod d = r)} \text{ that has a limit as } m \to \infty. \quad \#$$

$m = 0, 1, 2, \dots$