MATH4210: Financial Mathematics Tutorial 5

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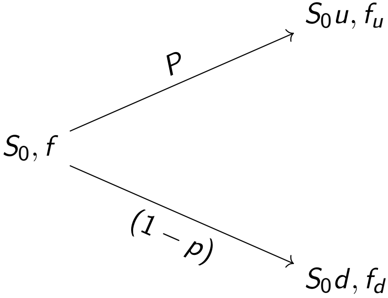
Binomial Tree Models

We have many inequality about options. Do we have equality?
Suppose the current stock price is S_0 and the option price with maturity T at time 0 is f. If we either have $S_T = S_0 u$ or $S_T = S_0 d$, for some $u > 1$ and $0 < d < 1$ and suppose the options price are f_u and f_d respectively with probability p and $1 - p$ respectively. Every term is given except f. Can we find f?



Binomial Tree Models

We consider two portfolios with initial wealth f.

1. Long ϕ_0 stock and long $x - \phi_0 S_0$ cash
2. Long 1 options

Then, then values of the portfolios are

$$\Pi_1(0) = (x - \phi_0 S_0) + \phi_0 S_0$$

$$\Pi_1(T) = \begin{cases} (x - \phi_0 S_0)e^{rT} + \phi_0 S_0 u & \text{if stock price move upwards} \\ (x - \phi_0 S_0)e^{rT} + \phi_0 S_0 d & \text{if stock price move downwards} \end{cases}$$

and we have

$$\Pi_2(0) = f$$

$$\Pi_2(T) = \begin{cases} f_u & \text{if stock price move upwards} \\ f_d & \text{if stock price move downwards} \end{cases}$$

Binomial Tree Models

Can we choose ϕ_0 and f such that $\Pi_1(T) = \Pi_2(T)$ and $\Pi_1(0) = \Pi_2(0)$? Yes, it suffices to solve

$$\begin{aligned} x &= f \\ (f - \phi_0 S_0)e^{rT} + \phi_0 S_0 u &= f_u \\ (f - \phi_0 S_0)e^{rT} + \phi_0 S_0 d &= f_d. \end{aligned}$$

Solve it, we have

$$\begin{aligned} \phi_0 &= \frac{f_u - f_d}{S_0(u-d)} \\ f = x &= e^{-rT}(f_u - \phi_0 S_0(u - e^{rT})) \\ &= e^{-rT}(q f_u + (1-q) f_d), \end{aligned}$$

where

$$q := \frac{e^{rT} - d}{u - d}$$

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Binomial Tree Models

If $q \in (0,1)$, define a probability measure $\mathbb{Q}$ (totally unrelated to p) that,

$$\begin{cases} \mathbb{Q}[S_T = S_0 u] & = \mathbb{Q}[f_T = f_u] = q \\ \mathbb{Q}[S_T = S_0 d] & = \mathbb{Q}[f_T = f_d] = 1 - q. \end{cases}$$

Then, we have

$$f = e^{-rT}\mathbb{E}^{\mathbb{Q}}[f_T] := \sum_{f_i} f_i \times \mathbb{Q}[f_t = f_i]$$

Note that we do Not necessarily have

$$p := \mathbb{P}(S_T = S_0 u) = q.$$

If this is the case, we call it risk neutral world.

Question

Given the current price of the underlying stock, $S(0) = 20$*. The stock price goes up and down by* $u = 1.2$ *and* $d = 0.67$*,respectively. The one period (continuously compounded) risk-less interest rate is* 10%*.*

a) Price a one period European call option with exercise price $K = 20$*.*

b) Price a two period European call option with exercise price $K = 20$ *in the tree.*

c) Price a two period European put option with exercise price $K = 20$ *in the tree.*

d) Suppose the put option is American. What is its price?

Answer

a)

1. $S(0) = 20,\ u = 1.2, d = 0.67, r = 0.1$
2. $S(0)u = 24,\ S(0)d = 13.4$
3. $q = \frac{e^{rT} - d}{u - d} \approx 0.82,\ 1 - q \approx 0.18$
4. $f_u = (S(0)u - K)^+ = 4,\ f_u = (S(0)d - K)^+ = 0$
5. $f = e^{-rT}(qf_u + (1-q)f_d) \approx e^{-0.1}(0.82 \times 4) = 2.9744$

Answer

$$
\begin{tikzcd}
 & & S_0u^2, f_{uu} \\
 & S_0u, f_u \arrow[ru] \arrow[rd] & \\
 & & S_0ud, f_{ud} \\
S_0, f \arrow[ruu] \arrow[rdd] & & \\
 & & S_0ud, f_{du} \\
 & S_0d, f_d \arrow[ru] \arrow[rd] & \\
 & & S_0d^2, f_{dd}
\end{tikzcd}
$$

Binomial Tree Models

Answer

$$\begin{tikzcd}
 & & S_0u^2 = 28.8 \\
 & S_0u = 24 \arrow[ur] \arrow[dr] & \\
 & & S_0ud = 16.08 \\
S_0 = 20 \arrow[uur] \arrow[ddr] & & \\
 & & S_0du = 16.08 \\
 & S_0d = 13.4 \arrow[ur] \arrow[dr] & \\
 & & S_0d^2 = 8.978
\end{tikzcd}$$

Binomial Tree Models

Answer

$$
\begin{tikzcd}
 & & f_{uu} = 8.8 \\
 & f_u \arrow[ur] \arrow[dr] & \\
 & & f_{ud} = 0 \\
f \arrow[uur] \arrow[ddr] & & \\
 & & f_{du} = 0 \\
 & f_d \arrow[ur] \arrow[dr] & \\
 & & f_{dd} = 0
\end{tikzcd}
$$

Binomial Tree Models

Answer

$$
\begin{tikzcd}
 & & f_{uu} = 8.8 \\
 & f_u = 6.5379 \arrow[ur] \arrow[dr] & \\
 & & f_{ud} = 0 \\
f = 4.8573 \arrow[uur] \arrow[ddr] & & \\
 & & f_{du} = 0 \\
 & f_d = 0 \arrow[ur] \arrow[dr] & \\
 & & f_{dd} = 0
\end{tikzcd}
$$

Binomial Tree Models

We have $f_d = 0$, does it contradict $C_E(t, T) > 0$ for all $t < T$?
No! We derive $C_E(t, T) > 0$ from the assumption that $\mathbb{P}(S(T) > K) > 0$. But at that node, we have $\mathbb{P}(S(T) > K) = 0$. Thus, this does not contradict our previous result.
Moreover, we still have $C_E(t, T_1) = 2.9744 < C_E(t, T_2) = 4.77$ with $T_1 < T_2$

Binomial Tree Models

Answer

$$
\begin{tikzcd}
 & & P_{uu} = 0 \\
 & P_u \arrow[ur] \arrow[dr] & \\
 & & P_{ud} = 3.92 \\
P \arrow[uur] \arrow[ddr] & & \\
 & & P_{du} = 3.92 \\
 & P_d \arrow[ur] \arrow[dr] & \\
 & & P_{dd} = 11.02
\end{tikzcd}
$$

Binomial Tree Models

Answer

$$
\begin{tikzcd}
 & & P_{uu} = 0 \\
 & P_u = 0.6346 \arrow[ur] \arrow[dr] & \\
 & & P_{ud} = 3.92 \\
P = 1.2319 \arrow[uur] \arrow[ddr] & & \\
 & & P_{du} = 3.92 \\
 & P_d = 4.6968 \arrow[ur] \arrow[dr] & \\
 & & P_{dd} = 11.022
\end{tikzcd}
$$

Obviously, our call-put parity holds in this case:

$$C_E(t, T) - P_E(t, T) = S(t) - Ke^{-r(T-t)}$$

For example, one has

$$C_d - P_d = -4.6968 = S_0 d - Ke^{-0.1} = 13.4 - 20e^{-0.1}$$

Binomial Tree Models

Answer

P

$P_u = \max(0.6346, 0)$

$P_{uu} = 0$

$P_{ud} = 3.92$

$P_d = \max(4.6968, 20 - 13.4)$

$P_{du} = 3.92$

$P_{dd} = 11.022$

Binomial Tree Models

Answer

$$
\begin{tikzcd}
 & & P_{uu} = 0 \\
 & P_u = 0.6346 \arrow[ur] \arrow[dr] & \\
 & & P_{ud} = 3.92 \\
P \arrow[uur] \arrow[ddr] & & \\
 & & P_{du} = 3.92 \\
 & P_d = 6.6 \arrow[ur] \arrow[dr] & \\
 & & P_{dd} = 11.022
\end{tikzcd}
$$

Binomial Tree Models

Answer

$$
\begin{tikzcd}
 & & P_{uu} = 0 \\
 & P_u = 0.6346 \arrow[ur] \arrow[dr] & \\
 & & P_{ud} = 3.92 \\
P = 1.5400 \arrow[uur] \arrow[ddr] & & \\
 & & P_{du} = 3.92 \\
 & P_d = 6.6 \arrow[ur] \arrow[dr] & \\
 & & P_{dd} = 11.022
\end{tikzcd}
$$

Binomial Tree Models

From this example, one can see that

Proposition

$f = e^{-2r\Delta t}(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd})$ *for a 2 period European option with period continuous compounding interest rate* r.

One can also deduce from induction that

Proposition

$$f = e^{-nr\Delta t}\sum_{i=0}^{n} C_i^n q^i (1-q)^{n-i} f_{u^i d^{n-i}} = e^{-nr\Delta t}\mathbb{E}^{\mathbb{Q}}[f_{n\Delta t}]$$

for an n *period European option with period continuous compounding interest rate* r.

Question

Suppose we still have $S_0 = K = 20$, $u = 1.2$, $d = 0.67, r = 0.1$, the price of a two period American put option is 1.5, can we obtain arbitrage profit?

Answer

We consider two portfolios:

1. *Long ϕ_0 stock and long $x - \phi_0 S_0$ cash*
2. *Long 1 options with price 1.5,*

where we choose

$$\phi_0 = \frac{P_u - P_d}{S_0(u-d)} = -0.5628, x = 1.54$$

.

Binomial Tree Models

Answer

Then, the values of this portfolio are

$$\Pi_1(0) = x = 1.54$$

$$\Pi_1(1) = \begin{cases} P_u & \textit{if stock price move upwards} \\ P_d & \textit{if stock price move downwards} \end{cases}$$

$$\Pi_2(0) = 1.5$$

$$\Pi_2(1) = \begin{cases} P_u & \textit{if stock price move upwards} \\ P_d & \textit{if stock price move downwards} \end{cases}$$

To obtain arbitrage profit, one suffices to short 1 *portfolio 1 and long* 1 *portfolio 2, i.e. long* $(-\phi_0)$ *stock, short* $(x - \phi_0 S_0)$ *cash, and long* 1 *put option.*

Question

Suppose you are given a two step binomial tree model with the following: $S_0 = 20, u = 1.2, d = 0.9, r = 0$. *Consider a two period Asian call option where the averaging is done over all three prices observed, i.e., the initial price, the price after one period, and the price after two periods, i.e.*

$$C_A(T, T) = (A(0, T) - K)^+,$$

where

$$A(0, T) := \mathbb{E}_t[S(t)] = \frac{1}{3}(S(0) + S(\Delta t) + S(2\Delta t))$$

Suppose the strike price is 20. *Find the initial price.*

Answer

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{1}{3}$$

Binomial Tree Models

Answer

$$
\begin{tikzcd}
 & & S_0uu = 28.8 \\
 & S_0u = 24 \arrow[ur] \arrow[dr] & \\
 & & S_0ud = 21.6 \\
S_0 = 20 \arrow[uur] \arrow[ddr] & & \\
 & & S_0du = 21.6 \\
 & S_0d = 18 \arrow[ur] \arrow[dr] & \\
 & & S_0dd = 16.2
\end{tikzcd}
$$

Answer

$$\begin{tikzcd}
 & & A_{uu} = 24.27 \\
 & S_0 u = 24 \arrow[ur] \arrow[dr] & \\
 & & A_{ud} = 21.87 \\
S_0 = 20 \arrow[uur] \arrow[ddr] & & \\
 & & A_{du} = 19.87 \\
 & S_0 d = 18 \arrow[ur] \arrow[dr] & \\
 & & A_{dd} = 18.07
\end{tikzcd}$$

Answer

$$
\begin{tikzcd}[column sep=huge]
 & & f_{uu} = 4.27 \\
 & f_u = 2.67 \arrow[ur] \arrow[dr] & \\
 & & f_{ud} = 1.87 \\
f = 0.89 \arrow[uur] \arrow[ddr] & & \\
 & & f_{du} = 0 \\
 & f_d = 0 \arrow[ur] \arrow[dr] & \\
 & & f_{dd} = 0
\end{tikzcd}
$$