THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4210 Financial Mathematics 2020-2021 T1
Tutorial Notes 4a

1. Suppose that we have the following 4 European call and put options with the same maturity T in the financial market:

Type	Strike Price	Price
Call	100	45
Call	110	40
Put	100	36
Put	110	42

Suppose that the continuous compounded interest rate is 5% in the market and the maturity time is $T = 1$, and assume the stock price is nonnegative. Construct a portfolio using some of options from the table and the Bank account to find an arbitrage profit.

Solution: We construct a portfolio by longing a and b call with strike 100 and 110, longing c and d put with strike 100 and 110 and long x cash. Then

$$\Pi(0) = aC_{E,100}(0,1) + bC_{E,110}(0,1) + cP_{E,100}(0,1) + dP_{E,110}(0,1) + x$$
$$\Pi(1) = a(S(1)-100)^+ + b(S(1)-110)^+ + c(100-S(1))^+ + d(110-S(1))^+ + xe^{0.05}$$

To find arbitrage opportunity, without loss of generality, we may assume $\Pi(0) = 0$, i.e. $x = -(aC_{E,100}(0,1) + bC_{E,110}(0,1) + cP_{E,100}(0,1) + dP_{E,110}(0,1))$ and we have $\Pi(1) \geq 0$ with $\mathbb{P}(\Pi(1) > 0) > 0$. By considering $\Pi(1) \geq 0$, we have

$$\Pi(1) = \begin{cases} -45e^{0.05}a - 40e^{0.05}b + (100 - S(1) - 36e^{0.05})c + (110 - S(1) - 42e^{0.05})d & \text{if } 0 \leq S(1) < 100 \\ (S(1) - 100 - 45e^{0.05})a - 40e^{0.05}b - 36e^{0.05}c + (110 - S(1) - 42e^{0.05})d & \text{if } 100 \leq S(1) < 110 \\ (S(1) - 100 - 45e^{0.05})a + (S(1) - 110 - 40e^{0.05})b - 36e^{0.05}c - 42e^{0.05}d & \text{if } S(1) \geq 110 \end{cases}$$
$$\geq 0$$

We must have $a + b \geq 0$, otherwise taking $S(1) \to \infty$ contradicts the third inequality. Since $\Pi(1)$ is piece-wisely linear in $S(1)$, it suffices to consider the equality at critical points, i.e. $S(1) = 0, 100$ and 110, as the minimum is only attained at the critical points. Thus, $\Pi(1) \geq 0$ can be rewritten as

$$\begin{cases} -45e^{0.05}a - 40e^{0.05}b + (100 - 36e^{0.05})c + (110 - 42e^{0.05})d & \geq 0 \\ -45e^{0.05}a - 40e^{0.05}b - 36e^{0.05}c + (10 - 42e^{0.05})d & \geq 0 \\ (10 - 45e^{0.05})a - 40e^{0.05}b - 36e^{0.05}c - 42e^{0.05}d & \geq 0 \\ a + b & \geq 0 \end{cases}$$

Since if (a, b, c, d) with $a + b > 0$ is a solution to the above inequality, $(a, -a, c, d)$ is also a solution and gives us an arbitrage opportunity. We may assume $a + b = 0$ at this stage. Thus, the above inequality is rewritten as

$$\begin{cases} -5e^{0.05}a + (100 - 36e^{0.05})c + (110 - 42e^{0.05})d & \geq 0 \\ -5e^{0.05}a - 36e^{0.05}c + (10 - 42e^{0.05})d & \geq 0 \\ (10 - 5e^{0.05})a - 36e^{0.05}c - 42e^{0.05}d & \geq 0 \end{cases}$$

If $a = 0$, one can deduce that $c = d = b = x = 0$, which is not arbitrage opportunity. Thus, we either have $a > 0$ or $a < 0$. Without loss of generality, by rescaling, we may assume $a = 1$ or -1. Suppose $a = 1$, then we have

$$\begin{cases} c & \geq -\dfrac{(110 - 42e^{0.05})d - 5e^{0.05}}{100 - 36e^{0.05}} \\ c & \leq \dfrac{(10 - 42e^{0.05})d - 5e^{0.05}}{36e^{0.05}} \\ c & \leq \dfrac{-42e^{0.05}d + 10 - 5e^{0.05}}{36e^{0.05}}, \end{cases}$$

which leads to a contradiction. Thus, we must have $a = -1$, i.e.

$$\begin{cases} c & \geq -\dfrac{(110 - 42e^{0.05})d + 5e^{0.05}}{100 - 36e^{0.05}} \\ c & \leq \dfrac{(10 - 42e^{0.05})d + 5e^{0.05}}{36e^{0.05}} \\ c & \leq \dfrac{-42e^{0.05}d - 10 + 5e^{0.05}}{36e^{0.05}}, \end{cases}$$

Solve it, we have

$$-1.4236 \cdots = -\frac{5e^{0.05}}{10 - 6e^{0.05}} \leq d \leq -\frac{50e^{-0.05} - 43}{12} = -0.3801 \ldots$$

We take $d = -1$, then $c = 1$. Also, we have $a = -1$, $b = 1$.

Now, check directly that if we short and long 1 call with strike 100 and 110, long and short 1 put with strike 100 and 110 and deposit 11 cash in the bank, then one can verify that the portfolio gives us an arbitrage opportunity.