MATH4210: Financial Mathematics Tutorial 3

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Forward

A Forward contract is a commitment to purchase at a future date a given amount of a commodity or an asset at a price agreed on today (forward price).

Forward

Suppose the continuous compounded interest rate is r, let $S(t)$ be the price of the product at time t, suppose you long a forward contract with forward price $F(0,T)$ signed at time 0 and matured at time T, then the portfolio is

$$\Pi(0) = 0$$
$$\Pi(T) = S(T) - F(0,T).$$

If you short the forward contract, then the portfolio becomes

$$\Pi(0) = 0$$
$$\Pi(T) = F(0,T) - S(T).$$

1. Two payoff functions sum to zero — zero-sum game.
2. Payoff can be negative — obligation.

The law of one price

1. If two portfolios are guaranteed to have the same value at a future time $T > t$ regardless of the state of the market at time T, then they must have the same value at time t.
2. For otherwise, one can long portfolio with lower cost and short the portfolio with higher cost to obtain arbitrage opportunity.

Forward

Question

Under no arbitrage opportunity assumptions and assume the continuous compounded interest rate is r, if the stock pays no dividend, show that $F(t, T) = S(t)e^{r(T-t)}$ for $t \geq T$.

Forward

Answer

Longing a forward contract and depositing $\$F(t, T)e^{-r(T-t)}$ *in the bank for exercising forward contract at time* 0 *will construct the portfolio*

$$\Pi_1(t) = F(t, T)e^{-r(T-t)}$$
$$\Pi_1(T) = F(t, T) + (S(T) - F(t, T)) = S(T).$$

Buying the stock a time t *will construct another portfolio*

$$\Pi_2(t) = S(t)$$
$$\Pi_2(T) = S(T).$$

Since $\Pi_1(T) = \Pi_2(T)$, *we must have* $\Pi_1(t) = \Pi_2(t)$, *i.e.*
$F(t, T) = S(t)e^{r(T-t)}$

Forward

Question

Suppose the stock price \$58 *at time* 0 *and the discrete compounded annual interest rate is* 3% *and it pays no dividend. Find the forward price of that stock matures in* 2 *years later.*

Answer

$F(0,2) = 58 \times 1.03^2 = 61.53$

Forward

Question

Suppose the stock pays a dividend d *at time* t*, where* $0 < t < T$*, find its forward price* $F(0, T)$*.*

Answer

Longing a forward contract and putting $\$F(0,T)e^{-rT} + de^{-rt}$ *money into the bank. The portfolio becomes*

$$\Pi_1(0) = F(0,T)e^{-rT} + de^{-rt}$$
$$\Pi_1(T) = F(0,T) + de^{r(T-t)} + (S(T) - F(0,T)) = S(T) + de^{r(T-t)}.$$

Buying the stock at time 0 *constructs a portfolio*

$$\Pi_2(0) = S(0)$$
$$\Pi_2(T) = S(T) + de^{r(T-t)}.$$

By the law of one price, we must have $\Pi_1(0) = \Pi_2(0)$, *i.e.* $F(0,T) = (S(0) - de^{-rt})e^{rT}$.

Forward

Question

Suppose the stock pay a dividend $d \times S(t)$ *at time* t*, where* $0 < t < T$ *and* $0 < d < 1$*, show its forward price* $F(0, T) = \dfrac{1}{1+d} S(0) e^{rT}$*.*

Answer

Do it in assignment 2.

Future

A futures contract: it is similar to a forward contract except that

- futures are traded on organized exchanges
- the fluctuations in the price of the underlying instrument are settled by daily payment between the parties

Two conditions are imposed

- $f(T, T) = S(T)$
- At each time step t_k, one compute the value $f(t_k, T)$ to make the value of a futures contract be zero after the payment $f(t_k, T) - f(t_{k-1}, T)$.

Similarly, if the stock pays no dividend, then $f(t, T) = F(t, T) = e^{r(T-t)}S(t)$.

Options

One party buys the RIGHT to buy or sell in the future from the other party. There are mainly two types of options:

- Call option: Holder has the right to buy (long call position); Writer has the obligation to sell (short call position).
- Put option: Holder has the right to sell (long put position); Writer has the obligation to buy (short put position).

There are different kind of options according to the policy in different places:

- European option: Can be exercised at maturity date only
- American option: Can be exercised at any day before the maturity date.
- Asian option, Barrier option, etc.

Options

Suppose the strike price is K and the stock price at time T is $S(T)$.
Call options: Payoff is $\max(S(T) - K, 0)$ (long), $-\max(S(T) - K, 0)$ (short).
Put options: Payoff is $\max(K - S(T), 0)$ (long), $-\max(K - S(T), 0)$ (short).

1. Two payoff functions sum to zero — zero sum game.
2. Payoff for long is nonnegative — right, not obligation
3. Payoff for short is nonpositive
4. Futures long = call options long + put options short (obligation)
5. Futures short = put options long + call options short (obligation)

Question

Consider a long call option with strike price 80 *with option price* 10 *mature at* 2 *years later. The current stock price is* $S(0) = 85$. *Suppose* $S(2) = 100$ *and the continuous compounded interest rate is* 3%, *find the payoff and profit (in future value).*

Answer

$$\begin{aligned} Payoff &= \max(S(T) - K, 0) = \max(100 - 80, 0) = 20 \\ Profit &= Payoff - option\ cost = 20 - 10e^{0.06} = 9.38 \end{aligned}$$

Options

Question

Show that the European put options with strike price K and maturity at time T satisfies $P_E(t,K) > Ke^{-r(T-t)} - S(t)$ for all $t < T$, where $S(t)$ is the stock price, r is the continuous compounded interest rate.

Options

Answer

Suppose there exists some $t_0 < T$ *such that* $P_E(t_0, K) \leq Ke^{-r(T-t_0)} - S(t_0)$. *We construct a portfolio by longing one put options and one stock and short* $Ke^{-r(T-t_0)}$ *cash. Then,* $\Pi(t_0) = P_E(t_0, K) + S(t_0) - Ke^{-r(T-t_0)} \leq 0$. *At time* T, *the portfolio becomes*

$$\begin{aligned}\Pi(T) &= (K - S(T))^+ + S(T) - K \\ &= \begin{cases} 0 & \text{if } S(T) \leq K \\ S(T) - K & \text{if } S(T) > K \end{cases} \\ &\geq 0\end{aligned}$$

Moreover, $\mathbb{P}(\Pi(T) > 0) = \mathbb{P}(S(T) > K) > 0$. *This leads to an arbitrage opportunity. Therefore,* $P_E(t, K) > Ke^{-r(T-t)} - S(t)$ *for all* $t < T$.

Options

Question

Suppose two put European options are identical except for the strike prices $0 < K_1 < K_2$*, show that*

$$0 < P_E(t, K_2) - P_E(t, K_1) < (K_2 - K_1)e^{-r(T-t)},$$

at any time t *before maturity* T*.*

Answer

Suppose $P_E(t_0, K_2) - P_E(t_0, K_1) \leq 0$ *for some* t_0. *Then, one can long one call with strike price* K_2 *and short one call with strike price* K_1. *Then, the portfolio at time* t_0 *is* $\Pi(t_0) = P_E(t_0, K_2) - P_E(t_0, K_1) \leq 0$. *At time* T, *the portfolio becomes*

$$\begin{aligned} \Pi(T) &= (K_2 - S(T))^+ - (K_1 - S(T))^+ \\ &= \begin{cases} K_2 - K_1 & \text{if } S(T) \leq K_1 \\ K_2 - S(T) & \text{if } K_1 < S(T) < K_2 \\ 0 & \text{if } S(T) \geq K_2 \end{cases} \\ &\geq 0 \end{aligned}$$

Moreover, $\mathbb{P}(\Pi(T) > 0) = \mathbb{P}(S(T) < K_2) > 0$. *This leads to an arbitrage opportunity. Therefore,* $P_E(t, K_2) - P_E(t, K_1) > 0$ *for all* $t < T$.

Options

Answer

Suppose $P_E(t_0, K_2) - P_E(t_0, K_1) \geq (K_2 - K_1)e^{-r(T-t_0)}$ *for some* $t_0 < T$. *Construct a portfolio at time* t*: long one put with strike price* K_1*, short one put with strike price* K_2 *and deposit* $\$(K_2 - K_1)e^{-r(T-t_0)}$ *in the bank. Then* $\Pi(t_0) = P_E(t_0, K_1) - P_E(t_0, K_2) + (K_2 - K_1)e^{-r(T-t_0)} \leq 0$.

$$
\begin{aligned}
\Pi(T) &= (K_1 - S(T))^+ - (K_2 - S(T))^+ + K_2 - K_1 \\
&= \begin{cases} 0 & \text{if } S(T) \leq K_1 \\ S(T) - K_1 & \text{if } K_1 < S(T) < K_2 \\ K_2 - K_1 & \text{if } S(T) \geq K_2 \end{cases} \\
&\geq 0
\end{aligned}
$$

Moreover, $\mathbb{P}(\Pi(T) > 0) = \mathbb{P}(S(T) > K_1) > 0$. *This leads to an arbitrage opportunity. Therefore,* $P_E(t, K_2) - P_E(t, K_1) < (K_2 - K_1)e^{-r(T-t)}$ *for all* $t < T$.