MATH 4210 - Financial Mathematics

Interest rate, derivatives, arbitrage

- Interest rate
  - Simple interest
  - Compounded interest (discretely or continuously)
  - (Net) Present Value, Loan formula, etc.

- Forward, Future,

$$F(t,T) = S(t)e^{r(T-t)}$$

- Arbitrage Opportunity

$$\Pi(0) = 0, \qquad \Pi(T) \geq 0, \qquad \mathbb{P}[\Pi(T) > 0] > 0.$$

- Vanilla options and No Arbitrage condition

Discrete time market

- Dynamic trading on a discrete time market

$$V_{t_k} = e^{rt_k}\left(V_{t_0} + \sum_{i=0}^{k-1} \phi_{t_i}\left(\widetilde{S}_{t_{i+1}} - \widetilde{S}_{t_i}\right)\right).$$

- Binomial tree model:
  - Replication strategy
  - Pricing under risk neutral probability measure
  - Multiple steps

Continuous time market: stochastic calculus

- Brownian motion
- Heat equation
- Stochastic Integration, Itô's Lemma
  - Memorise the formulas, know how to apply these formulas.
  - The proofs are not required
- Black-Scholes model

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad S_t = S_0 \exp\big((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\big).$$

Continuous time market: pricing, hedging

- Dynamic trading in continuous time market

$$d\Pi_t \;=\; \phi_t dS_t \;+\; (\Pi_t - \phi_t S_t) r dt.$$

- Replication strategy leading to Black-Scholes PDE:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0, \\ u(T,s) = g(s). \end{cases}$$

- Probabilistic representation under risk neutral probability measure

$$u(0,S_0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} g(S_T)\right].$$

- Deduce explicit Black-Scholes formula for Call/Put options, Carr-Madan formula.

Black-Scholes pricing

Recall that the price of a European option with payoff $g(S_T)$ is the solution of PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0, \\ u(T, s) = g(s). \end{cases}$$

Or equivalently, it is given by

$$u(0, S_0) \;=\; \mathbb{E}^{\mathbb{Q}}\big[e^{-rT} g(S_T)\big],$$

where $\mathbb{Q}$ is the risk neutral probability, under which the stock price follows:

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma B_T^{\mathbb{Q}}},$$

for some $\mathbb{Q}$-Brownian motion $B^{\mathbb{Q}}$.

Monte Carlo method, an introduction

Let X be a random vector taking value in $\mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}$, $(X_k)_{k\geq 1}$ be a sequence of i.i.d. random vectors with the same distribution of X. Then the Monte-Carlo estimator of $\mathbb{E}[f(X)]$ is given by

$$\overline{Y}_n := \frac{1}{n}\sum_{k=1}^{n} Y_k \quad \text{where} \quad Y_k := f(X_k).$$

Monte Carlo method, an introduction

Let Y be a random variable, $(Y_k)_{k\geq 1}$ be a sequence of i.i.d. random variables with the same distribution of Y, and

$$\overline{Y}_n := \frac{1}{n}\sum_{k=1}^{n} Y_k.$$

Theorem 2.1 (Law of large number)

Assume that $\mathbb{E}[|Y|] < \infty$, *then*

$$\overline{Y}_n \to \mathbb{E}[Y] \text{ almost surely as } n \to \infty.$$

Theorem 2.2 (Central Limit Theorem)

Assume that $\mathbb{E}[|Y|^2] < \infty$, *then*

$$\sqrt{n}\frac{\overline{Y}_n - \mathbb{E}[Y]}{\sqrt{\mathrm{Var}[Y]}} \Rightarrow \mathcal{N}(0,1) \text{ in distribution.}$$

Monte Carlo method, an introduction

- $\xi_n \Rightarrow N(0,1)$ in distribution means that

$$\mathbb{P}\big[\xi_n \in [a,b]\big] \;\rightarrow\; \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx.$$

- This implies that for n large enough

$$\begin{aligned} p(R) := \int_{-R}^{R} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx &= \mathbb{P}\Big[\sqrt{n}\frac{\overline{Y}_n - \mathbb{E}[Y]}{\sqrt{\mathrm{Var}[Y]}} \in [-R,R]\Big] \\ &= \mathbb{P}\Big[\mathbb{E}[Y] \in \big[\overline{Y}_n - \frac{\sqrt{Y}}{n}R, \overline{Y}_n + \frac{\sqrt{Y}}{n}R\big]\Big] \end{aligned}$$

We then obtain the confidence interval (with a confidence level $p(R)$):

$$\Big[\overline{Y}_n - \frac{\sqrt{\mathrm{Var}[Y]}}{\sqrt{n}}R,\ \overline{Y}_n + \frac{\sqrt{\mathrm{Var}[Y]}}{\sqrt{n}}R\Big].$$

Remark: for $R = 2$, one has $p(R) \approx 95\%$.

Monte Carlo method, an introduction

- In practice, we use the following estimator to estimate $\mathrm{Var}[Y]$:

$$s_n^2 \;:=\; \frac{1}{n}\sum_{k=1}^{n}(Y_k - \overline{Y}_n)^2 \;=\; \frac{1}{n}\sum_{k=1}^{n} Y_k^2 - \left(\overline{Y}_n\right)^2.$$

- : In summary

1. Simulate an i.i.d. sequence $(X_k)_{k\geq 1}$, let $Y_k := f(X_k)$.
2. The estimator:

$$\overline{Y}_n \;:=\; \frac{1}{n}\sum_{k=1}^{n} Y_k.$$

3. The confidence interval:

$$\Big[\overline{Y}_n - \frac{s_n}{\sqrt{n}}R,\ \overline{Y}_n + \frac{s_n}{\sqrt{n}}R\Big].$$

Simulation of random variables

• We accept that one has the generator for uniform distribution $\mathcal{U}[0,1]$.

• Inverse method: let $F : \mathbb{R} \to [0,1]$ be the distribution function of a random variable X, $U \sim \mathcal{U}[0,1]$, then

$$X \sim F^{-1}(U) \text{ in distribution.}$$

• Transformation method (Box-Muller, not required): let U and V be two independent random variables of uniform distribution on $[0,1]$, let

$$X := \sqrt{-2\log(U)}\cos(2\pi V) \quad \text{and} \quad Y := \sqrt{-2\log(U)}\sin(2\pi V).$$

Then X and Y are two independent random variable of Gaussian distribution $N(0,1)$.

Simulation of a brownian motion

Let B be a Brownian motion, using the independent and stationary increment property of the Brownian motion, we use the following algorithme to simulate a Brownian motion B at finite time instants $0 = t_0 < t_1 < \cdots < t_n = T$:

- Simulate a sequence of i.i.d. random variables $(Z_k)_{k=1,\cdots,n}$ of distribution $N(0,1)$.
- Let $B_{t_0} = 0$ and then the iteration:

$$B_{t_{k+1}} \;=\; B_{t_k} \;+\; \sqrt{t_{k+1} - t_k}\; Z_{k+1}.$$

Variance reduction

• Main idea: to estimate $\mathbb{E}[f(X)]$, one find another function $g : \mathbb{R}^d \to R$ such that

$$\mathbb{E}\big[g(X)\big] = \mathbb{E}\big[f(X)\big] \quad \text{and} \quad \mathrm{Var}\big[g(X)\big] < \mathrm{Var}\big[f(X)\big].$$

• Example (Antithetic method): let $S_T := S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma B_T)$, we define the antithetic variable

$$A(B_T) := -B_T \sim B_T \quad \text{and} \quad A(S_T) := S_0 \exp((r - \sigma^2/2)T + \sigma A(B_T)),$$

then

$$\mathbb{E}\big[e^{-rT} f(S_T)\big] \;=\; \mathbb{E}\big[e^{-rT} g(S_T)\big] \text{ for } g(S_T) := \frac{f(S_T) + f(A(S_T))}{2}.$$

• There are many other methods allowing to find functions $g : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}\big[g(X)\big] = \mathbb{E}\big[f(X)\big]$, further analysis are need to check if one has $\mathrm{Var}\big[g(X)\big] < \mathrm{Var}\big[f(X)\big]$.

Finite difference method

- The Black-Scholes PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0, \\ u(T, s) = f(s). \end{cases}$$

- Time discretization of the interval $[0, T]$:

$$0 = t_0 < t_1 < \cdots t_n = T, \quad \text{where} \;\; t_k := k\Delta t, \quad \Delta t := \frac{T}{n}.$$

- Space discretization of the interval $[R_1, R_2]$: for two integers $r_1 < r_2$,

$$R_1 = s_{r_1} < s_{r_1+1} < \cdots < s_{r_2-1} < s_{r_2} = R_2.$$

- Numerical solution $\hat{u}^h(t_k, s_i)$ on the grid (t_k, s_i), let us denote

$$\hat{u}_i^k \;:=\; \hat{u}^h(t_k, s_i).$$

Finite difference method

- Numerical solution $\hat{u}^h(t_k, s_i)$ on the grid (t_k, s_i), let us denote

$$\hat{u}_i^k \;:=\; \hat{u}^h(t_k, s_i), \qquad \Delta s = s_{i+1} - s_i.$$

- Approximate the derivatives:

$$\frac{\partial u}{\partial t}(t_k, s_i) \;\approx\; \frac{\hat{u}_i^k - \hat{u}_i^{k-1}}{\Delta t}, \qquad \frac{\partial u}{\partial s}(t_k, s_i) \;\approx\; \frac{\hat{u}_{i+1}^k - \hat{u}_i^k}{\Delta s},$$

and

$$\frac{\partial^2 u}{\partial s^2}(t_k, s_i) \;\approx\; \frac{\hat{u}_{i+1}^k - 2\hat{u}_i^k + \hat{u}_{i-1}^k}{\Delta s^2}.$$

- Plugging the above expression into the PDE, it leads to

$$\hat{u}_i^{k-1} \;=\; A_i u_{i+1}^k + B_i u_{i-1}^k + C_i u_i^k.$$

Finite difference method

- The numerical scheme:

$$\hat{u}_i^{k-1} \;=\; A_i\hat{u}_{i+1}^k + B_i\hat{u}_{i-1}^k + C_i\hat{u}_i^k,$$

where

$$A_i = rs_i\frac{\Delta t}{\Delta s} + \frac{1}{2}\sigma^2 s_i^2\frac{\Delta t}{\Delta s^2}, \qquad B_i = \frac{1}{2}\sigma^2 s_i^2\frac{\Delta t}{\Delta s^2},$$

and

$$C_i = 1 - A_i - B_i - r\Delta t.$$

- The boundary condition $R_1 = 0$:

$$u(t, R_1) = e^{-r(T-t)}f(0) \quad\Longrightarrow\quad \hat{u}_{r_1}^k = e^{-r(T-t)}f(0);$$

on the right hand side $R_2 = 2S_0$, for call option $f(s) = (s-K)_+$,

$$\partial_s u(t, R_2) = 1, \quad\Longrightarrow\quad \hat{u}_{r_2}^k = \hat{u}_{r_2-1}^k + \Delta s.$$

Finite difference method

Theorem 2.3 (Not required)

Assume that $C_i \geq 0$ for all i. Then one has

$$\hat{u}^h \longrightarrow u \quad \text{as} \quad (\Delta t, \Delta s) \longrightarrow 0.$$

Finite difference method

• We can rewrite the above scheme as follows:

$$\begin{aligned}\hat{u}^h(t_k, s_i) &= \mathbb{T}_h[\hat{u}^h(t_{k+1}, \cdot)](t_k, s_i) \\ &= A_i \hat{u}^h(t_{k+1}, s_{i+1}) + B_i \hat{u}^h(t_{k+1}, s_{i-1}) + C_i \hat{u}^h(t_{k+1}, s_i),\end{aligned}$$

and one has the so-called consistency condition, i.e.

$$\frac{\mathbb{T}_h[u(t_{k+1}, \cdot)](t_k, s_i) - u(t_k, s_i)}{\Delta t} \longrightarrow \partial_t u + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} u + rs\partial_s u - ru,$$

as $(\Delta t, \Delta x) \to 0$.

• More generally, for other numerical schemes satisfying the consistency condition, one may also prove the convergence.

A binomial tree scheme

The binomial tree method is given by

$$\begin{aligned}\hat{u}^h(t,s) &= \mathbb{T}_h[\hat{u}^h(t+\Delta t,\cdot)](t,s)\\ &= e^{-r\Delta t}\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\ \hat{u}^h(t+\Delta t, se^{\sigma\sqrt{\Delta t}})\\ &\quad + e^{-r\Delta t}\frac{e^{\sigma\sqrt{\Delta t}}-e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\ \hat{u}^h(t+\Delta t, se^{-\sigma\sqrt{\Delta t}}).\end{aligned}$$

One can check directly that it satisfies the consistency condition:

$$\frac{\mathbb{T}_h[u(t_{k+1},\cdot)](t,s)-u(t,s)}{\Delta t} \longrightarrow \partial_t u + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} u + rs\partial_s u - ru,$$

as $\Delta t \to 0$.

A binomial tree scheme

Theorem 2.4 (Not required)

For the binomial tree scheme, one has

$$\hat{u}^h \longrightarrow u \quad \text{as} \quad \Delta t \longrightarrow 0.$$