THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH5022 Theory of Partial Differential Equations, 2nd Term 2016-17

Background of the course

1 L^2 Theory

Consider the Dirichlet problem on the 2nd order linear inhomogeneous elliptic PDE

$$-D_j(a^{ij}D_iu) + b^iD_iu + cu = f + D_if^i \quad \text{in } \Omega \tag{1}$$
$$u = g \quad \text{on } \partial\Omega, \tag{2}$$

with all coefficient functions in $L^\infty(\Omega)$. L^2 theory is to study

- Existence of *weak solutions* in $H^1(\Omega)$. The proof is based on the Lax-Milgram Theorem in the functional analysis. The result roughly says that there exists a weak solution $u \in H^1(\Omega)$ if

$$f + D_if^i \in H^{-1}(\Omega), \quad g \in H^1(\Omega). \tag{3}$$

- L^∞ estimate on weak solutions. It can be based on the DeGiorgi or Moser iteration. For instance, one may show

$$\sup_\Omega |u| \le \sup_{\partial\Omega} u^+ + C(\|f\|_{L^{\frac{np}{n+p}}(\Omega)} + \|f^i\|_{L^p(\Omega)})|\Omega|^{\frac{1}{n}-\frac{1}{p}}. \tag{4}$$

- Regularity. This is to show that, if data are more *regular*, for instance, $a^{ij} \in W^{1,\infty}$, $g \in H^2$, then solutions are more *regular* correspondingly, e.g. $u \in H^2(\Omega)$. The proof usually includes two steps:

  - Interior regularity: Estimate u over any compact subset of Ω.
  - Boundary regularity: Estimate u near any boundary point.

  Combining both gives the global regularity estimate in $H^2(\Omega)$.

2 Schauder Theory

The topic is concerned with the existence and regularity of *classical solutions* for either linear or nonlinear (for instance, quasilinear) elliptic PDEs.

- The technique of Schauder Theory can be understood in terms of the a priori $C^{2,\alpha}$ estimate for the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega = \mathbb{R}^n. \tag{5}$$

  That is to estimate norms

$$\sup_\Omega |u|, \quad \sup_\Omega |Du|, \quad \sup_\Omega |D^2 u|, \quad [D^2 u]_{\alpha,\Omega}, \tag{6}$$

  in terms of given data source term $f \in C^\infty$, with compact support, for instance, step by step. The key step is Hölder estimate on $D^2 u$. Extensions of the Poisson equation can be made to

  - Consider the a little general form

$$-a^{ij} D_{ij} u = f. \tag{7}$$

  - Consider the half space domain

$$\Omega = \mathbb{R}^n_+. \tag{8}$$

- In case Ω is a bounded domain in $\mathbb{R}^n$, consider

$$-a^{ij} D_{ij} u + b^i D_i u + cu = f \quad \text{in } \Omega \tag{9}$$

$$u = \varphi \quad \text{on } \partial\Omega, \tag{10}$$

  where all data are regular, for instance, at least C^α. The Schauder estimates still can be carried out in the way that

  - Interior estimate: It is to bound $\|u\|_{2,\alpha;\Omega'}$ for any compact subset Ω'.
  - Global estimate: It is to combine the interior estimate with the boundary estimate to obtain

$$\|u\|_{2,\alpha;\Omega} \le C(\|f\|_{\alpha;\Omega} + \|\varphi\|_{2,\alpha;\Omega} + \|u\|_{0;\Omega}). \tag{11}$$

    Note that $\|u\|_{0;\Omega} = \sup_\Omega |u|$ can be estimated by maximum principle of classical solutions.

- Existence of classical solutions follows by the following steps:

  - Approximate data smoothly.
  - Show the existence of weak solutions basing on L^2 theory.
  - Show the existence of classical solutions basing on the compactness argument Ascoli-Arzela Theorem for approximate solutions with uniform $C^{2,\alpha}$ bounds.

3 L^p Theory

For instance, consider

$$-a^{ij}D_{ij}u + b^iD_iu + cu = f \quad \text{in } \Omega \tag{12}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{13}$$

The theory is to find *strong solutions* (exists uniquely) in $W^{2,p}(\Omega)$ for $1 < p < \infty$. In order for such L^p theory to be applicable, besides the standard assumptions on ellipticity as well as boundedness of coefficients a^{ij}, b^i and c ($c \geq 0$), an essential condition is

$$a^{ij} \in C(\overline{\Omega}), \tag{14}$$

i.e. the coefficients of second order terms must be uniformly continuous on Ω. The a priori estimate is to obtain

$$\|u\|_{W^{1,p}(\Omega)} \leq \frac{C}{\lambda}\|f\|_{L^p(\Omega)}, \tag{15}$$

where $\lambda > 0$ is the lower bound of (a^{ij}), and C particularly depends on the continuity norms of a^{ij}. An understanding of the necessity of the condition $a^{ij} \in C(\overline{\Omega})$ for the whole theory working well is that one may compare the equation to the Poisson equation in the way

$$-a^{ij}(x_0)D_{ij}u = \overline{f}, \tag{16}$$

for an arbitrarily fixed point x_0 in Ω. Therefore, L^p estimates on the Poisson equation is a key. For that, analytical tools include Marcinkiewicz's Interpolation Theorem as well as Decomposition Lemma associated with a nonnegative L^1 function on $\mathbb{R}^n$ due to Calderón and Zygmund. Note that the procedures for $W^{2,p}$ estimates are quite similar to those for the Schauder estimates.

4 How DeGiorge-Nash-Moser meet Hölder

In both Schauder Theory and L^p Theory discussed before, it is required that a^{ij} are continuous on $\overline{\Omega}$. Otherwise, two theories do not work. This makes the linear theory of elliptic equations not directly applicable for the nonlinear equations. The breakthrough has been made by

- De Giorgi (1957): He obtained the Höder estimates for

  $$Lu = f \quad \text{in } \Omega \tag{17}$$

  for a divergence-form elliptic operator, with

  $$a^{ij} \in L^\infty(\Omega). \tag{18}$$

- Nash (1958): He obtained the similar Höder estimates for the parabolic equation

$$u_t + Lu = f \tag{19}$$

with $a^{ij} \in L^\infty(\Omega)$ only.

- Moser (1960): He gave a new proof of Höder estimates by DeGiorgi-Nash, and developed the Harnack inequalities in the same framework.

This becomes a classical approach by DeGiorgi-Nash-Moser for regularity of weak solutions to elliptic PDEs with L^∞ coefficients, and makes it possible to apply the linear theory to the quasilinear theory.

The main goal of DeGiorgi-Nash-Moser estimates on the elliptic equation with L^∞ coefficients is to derive C^α estimates from L^∞ bounds. For instance, the procedure includes

- Interior estimates:

$$\sup_{B_{\theta R}} u \leq C(\frac{1}{|B_R|}\int_{B_R}(u^+)^p\,dx)^{\frac{1}{p}} \tag{20}$$

for any $1 < p < 1$, $0 < \theta < 1$. It is a place where DeGiorgi-Nash iteration or Moser iteration is useful.

- Combine the interior estimate with the boundary estimates to show $u \in C^\alpha(\Omega)$.

5 Second Order Quasilinear Elliptic PDEs

Consider

$$-a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du) = f \quad \text{in } \Omega \tag{21}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{22}$$

where Ω is a bounded domain, and all data a^{ij}, b, φ are regular. Note that the linear theory cannot be directly applied due to the nonlinearities in the equation. The new theory for the above quasilinear elliptic PDE was pioneered by Leray and Schauder in the 1930s. The key part of the Leray-Schauder Existence Theory is the combination of the argument by Leray-Schauder Fixed Point Theorem and the a priori estimates. The technique of DeGiorgi-Nash-Moser indeed makes it possible to obtain the necessary estimates on solutions in the nonlinear as in the linear case such as the maximum principle, so the fixed point argument can be adopted to lead to a solution.

Formally, the existence theory due to Leray-Schauder can be carried out in the following way. To study the original Dirichlet problem, we consider a family of related problems of the same type in terms of a parameter $t \in [0,1]$

$$P_t(u) = f \quad \text{in } \Omega \tag{23}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{24}$$

Here $t = 1$ reduces to the original equation $P_1(u) = 0$ and the problem in case $t = 0$ is usually a solvable linear elliptic PDE, for instance, we may take $P_0(u) = \Delta u$ and thus

$$P_t(u) = tP_1(u) + (1 - t)\Delta u, \quad 0 \leq t \leq 1. \tag{25}$$

Define

$$S = \{0 \leq t \leq 1 : \text{ the problem above is solvable}\}. \tag{26}$$

It is clear to see $S \neq \emptyset$. The a priori $C^{2,\alpha}$ estimates (*Schauder estimates*) can be used to show that S is both open and closed, and then $S = [0, 1]$, meaning that the original problem is solvable. The existence theorem works obviously for the inhomogeneous boundary data $u|_{\partial\Omega} = \varphi \in C^{2,\alpha}(\partial\Omega)$ and further for $\varphi \in C^0(\partial\Omega)$ by the approximation argument. In sum, the Leray-Schauder Existence Theorem says that *the problem* $P_1(u) = f$ *in* Ω, $u = \varphi$ *on* $\partial\Omega$, *where* $f \in C^\alpha$ *and* $\varphi \in C^0$, *has a unique solution* $u \in C^{2,\alpha}(\Omega)$.

As mentioned, one of key parts in the Leray-Schauder Theory is $C^{2,\alpha}$ estimate, particularly $C^{1,\alpha}$ estimate due to the quasilinear form. We include the following general strategy for obtaining $C^{1,\alpha}$ bounds:

- Estimate $\sup_\Omega |u|$ in terms of all data f and/or φ.
- Estimate $\sup_\Omega |Du|$ in terms of $\sup_{\partial\Omega} |Du|$.
- Estimate $\sup_{\partial\Omega} |Du|$ in terms of $\sup_\Omega |u|$. The estimate is based on a barrier construction, and thus geometric properties of the boundary $\partial\Omega$ are necessarily made more precise.
- Estimate $[Du]_{\alpha,\Omega}$ in terms of obtained bounds. It is a place how DeGiorge-Nash-Moser meet Hölder.

We note that the above procedure can be made in the relatively easier way when it is of the divergence form than in the non-divergence form. In the non-divergence form, the proof is due to

- Krylov-Safonov (1980): a first proof.
- Trudinger (later): a simplified proof.

Here in both proofs, the Alexandroff Maximum Principle is a key.

6 Fully Nonlinear Elliptic PDEs

They take the form of

$$F(x, u, Du, D^2u) = 0. \tag{27}$$

Here are a few examples.

- Monge-Ampère equation:
$$\det\left(D^2 u\right) = f(x) > 0. \tag{28}$$

- Gauss curvature equation:
$$\det(D^2 u) = K(x)(1 + |Du|^2)^{\frac{n+2}{2}}. \tag{29}$$

- Bellman equation arising from control theory:
$$\sup_k (L^k u - f^k) = 0, \tag{30}$$
  where L^k are linear elliptic operators.

The study of the fully nonlinear elliptic PDEs needs to use the technique in nonlinear analysis, for instance, Implicit Existence Theorem. Here, we may refer to the monograph by Caffarelli and Cabré.