Topics in Optimization I.

Part I: Static Optimization. $\inf_{x \in K} f(x)$.

$$f: \mathbb{R}^d \to \mathbb{R}, \quad K \subseteq \mathbb{R}^d.$$

Part II: Dynamic Optimization.

$$\inf_{X=(X_t)_{t=0,1,\dots,T}} \underbrace{J(X)}_{\hookrightarrow \sum_{t=0}^{T} L(t, X_t)}.$$

- Midterm Exam.
- Reserch paper $\rightarrow$ presentation, $\rightarrow$ report.

I. Static Optimization, $\rightarrow$ 1. Existence.

$$\boxed{(P): \inf_{x \in K} f(x).} \rightarrow \min_{x \in K} f(x).$$

2. (under constraints): necessary condition. (Kuhn-Tucker)

$\hookrightarrow K = \{x \in \mathbb{R}^n : g_i(x) \le 0, \ h_j(x) = 0\}$

$i = 1, \cdots, m$, $j = 1, \cdots, \ell$.

$f: \mathbb{R}^n \to \mathbb{R}$.

$K \subseteq \mathbb{R}^n$.

↳ nonempty.

3. Convex problem / duality approach.

4. Numerical methods.

1. Existence of the solution.

Definition: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function and $K \subseteq \mathbb{R}^n$. nonempty.

we call the infimum of f on K is the real value $\ell \in [-\infty, +\infty)$

s.t. ① $\ell \leq f(x)$, $\forall x \in K$.

② $\exists (x_n)_{n \geq 1} \subseteq K$. s.t. $f(x_n) \to \ell$.

↳ a minimization sequence.

We denote $\ell = \inf_{x \in K} f(x)$.

Remark: $\inf_{x \in K} f(x)$ always exists and it is unique.

Definition: If $\inf_{x \in K} f(x) > -\infty$, and there exists $x^* \in K$ s.t.

$f(x^*) = \inf_{x \in K} f(x)$. Then we say x^* is a solution to (P).

In this case, we also write $\boxed{\min_{x\in K} f(x) = f(x^*)}$

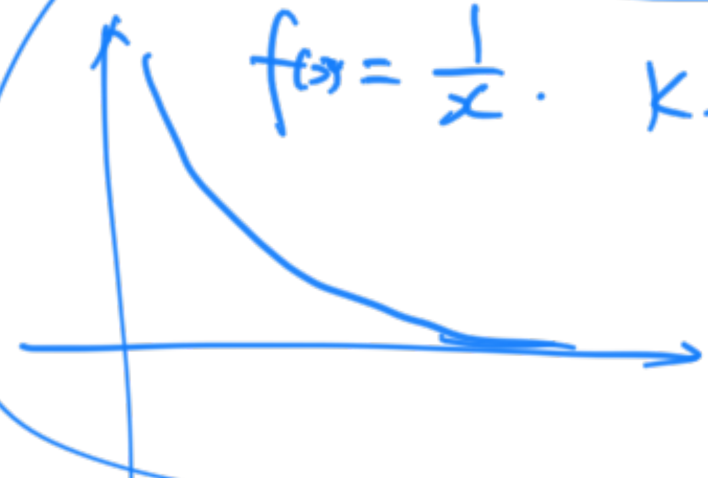
Example. (non-existence.) $n=1$, $f(x) = x$. $K = (0, 1)$.

Definition: We say a function $f: \mathbb{R}^n \to \mathbb{R}$ is coersive

If $\lim_{\|x\| \to +\infty} f(x) = +\infty$

not coersive.

$f(x) = \frac{1}{x}$. $K = (0, +\infty)$.



Example: $f(x) = x^T A x + b^T x + c$.

is coersive if A is definitely positive. matrix.

Proposition: If $f_i: \mathbb{R} \to \mathbb{R}$ is bounded from below, and coersive for each $i = 1, \dots, n$.

Then. $f(x_1, \dots, x_n) := \sum_{i=1}^{n} f_i(x_i)$. is coersive.

Proof: By adding a constant, we can assume that f_i are positive.

Then, Let $(x_n)_{n \to +\infty}$ s.t. $\|x_n\| \to +\infty$, one has some $i \in \{1, \dots, n\}$

$|x_n^i| \to +\infty$

$\left(\|x\| := \sqrt{\sum_i^n x_i^2} \right)$

$\Rightarrow f(x_n) \geq f_i(x_n^i) \longrightarrow +\infty$, as $n \to +\infty$. #

Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous and coersive.

$K \subseteq \mathbb{R}^n$ is nonempty.

Moreover, one of the two conditions holds.

① K is compact, ② K is closed and f is coersive.

Then, (P) has a solution. (i.e. $\exists\, x^* \in K$ s.t. $f(x^*) = \inf\limits_{x \in K} f(x)$.)

Proof: Case ①: K is compact:

Let us take a sequence of minimization $(x_n)_{n \geq 1} \subseteq K$. i.e. $\lim f(x_n) = \inf\limits_{x \in K} f(x)$.

$\Rightarrow \exists\, n_k$. s.t. $x_{n_k} \longrightarrow x^* \in K$.

$\Rightarrow f(x^*) = \lim\limits_{k \to +\infty} f(x_{n_k}) = \inf\limits_{x \in K} f(x)$.

Case ②: K is closed, and f is coersive.

Let. $\inf_{x \in K} f(x) < M < +\infty$.

Then: $\exists\, L > 0$ s.t. $f(x) \geq M$, $\forall \|x\| \geq L$. (coersive property of f)

$\Rightarrow$. $\inf_{x \in K} f(x) = \inf_{x \in K \cap \{x : \|x\| \leq L\}} f(x)$.

$\hookrightarrow$ bounded, closed $\Rightarrow$ compact.

$\Rightarrow$. $\exists\, x^* \in K \cap \{x : \|x\| \leq L\}$. s.t. $f(x^*) = \inf_{x \in K} f(x)$. $\cancel{\times}$

Proposition. Let K be a open and bounded subset of $\mathbb{R}^n$.

and f is continuous on $\overline{K}$ (closure of K.) $:= \{x : \exists (x_n) \subseteq K$ s.t. $x_n \to x\}$

and there exists $x_0 \in K$ s.t. $f(x_0) < f(x)$, $\forall x \in \partial K := \overline{K} \setminus K$.

Then. (P) has a solution $x^* \in K$.

Proof: Let us consider $\inf_{x \in \overline{K}} f(x)$, which has a solution $\overline{x}^* \in \overline{K}$.

If $x^* \in K$, then x^* is a solution to $\inf\limits_{x \in \bar{K}} f(x)$.

If $x^* \in \partial K$, then $x_0 \in K$ satisfies $f(x_0) \le f(x^*) = \inf\limits_{x \in \bar{K}} f(x)$

$\Longrightarrow$ x_0 is a solution to $\inf\limits_{x \in \bar{K}} f(x) = \inf\limits_{x \in K} f(x)$. #

2. Necessary condition of the optimal solution x^*.

2.1. Euler condition.

Theorem. Let $K \subseteq \mathbb{R}^n$ be an open set, $f: \mathbb{R}^n \to \mathbb{R}$ is in class C^1.
Assume that (P) has a solution $x^* \in K$.

Then $\nabla f(x^*) = 0$.

Proof. Let $e \in \mathbb{R}^n$, then for $\varepsilon > 0$ small enough, $\boxed{x_\varepsilon := x^* + \varepsilon e \in K}$

$\Longrightarrow$ $\lim\limits_{\varepsilon \downarrow 0} \dfrac{f(x^*+\varepsilon e) - f(x^*)}{\varepsilon} = \langle \nabla f(x^*), e \rangle \ge 0$, $\forall e$.

Since $f(x^*) \le f(\underbrace{x^* + \varepsilon e}_{\in K})$

$\Longrightarrow$ $\langle \nabla f(x^*), -e \rangle \ge 0$

$$\Longrightarrow \quad \nabla f(x^*) = 0. \qquad \#$$

2.2. Kuhn-Tucker Theorem.

→ $K = \{x \in \mathbb{R}^n : \underline{g_i(x) \le 0}, \ i = 1, \cdots, \ell. \ (i \in I.) \quad h_j(x) = 0, \ j = 1, \cdots, m. \ (j \in J.)\}$ K is a closed set.

→ $I = \{1, \cdots, \ell\}, \quad J = \{1, \cdots, m\}$

→ $f : \mathbb{R}^n \to \mathbb{R}, \quad g_i : \mathbb{R}^n \to \mathbb{R}, \quad h_j : \mathbb{R}^n \to \mathbb{R}$

→ $f, g_i, h_j \in \underline{C^1}.$

Theorem. Let x^* be a solution to (P). $\left(\longrightarrow \inf_{x \in K} f(x). \right)$

Then, there exists $p_0 \ge 0$, $p \in \mathbb{R}^\ell_+$, $q \in \mathbb{R}^m$.

s.t.

$$\begin{cases} p_0 \nabla f(x^*) + \sum_{i \in I} p_i \nabla g_i(x^*) + \sum_{j \in J} q_j \nabla h_j(x^*) = 0 \\ (p_0, p, q) \ne 0 \end{cases}$$

$(p_0, p, q) \neq 0$.

$$\sum_{i \in I} p_i g_i(x^*) = 0 \Rightarrow p_i g_i(x^*) = 0, \ \forall i, \Rightarrow \text{either } p_i = 0 \text{ or } g_i(x^*) = 0$$

$$\left.\begin{array}{l} p_i \geq 0 \\ g_i(x^*) \leq 0 \end{array}\right\} \Rightarrow p_i g_i(x^*) \leq 0, \ \forall i \in I$$

Discussions. ①.

②. p_0 could possibly equals to 0.

Example: $\min x$. $x^2 = 0$.

$f(x) = x$.

$h(x) = x^2$.

$\Rightarrow K = \{x : x^2 = 0\} = \{0\}$

$x^* = 0$.

$\Rightarrow \nabla f(x^*) = 1$, $\nabla h(x^*) = 2x^* = 0$.

$\Rightarrow$ to ensure that $p_0 \nabla f(x^*) + q \cdot \nabla h(x^*) = 0$

we need to set $p_0 = 0$, $q \neq 0$ | so that $(p_0, q) \neq 0$)

③ Formally, $\inf_{\substack{g(x)\le 0\\ h(x)=0}} f(x). \Leftrightarrow \inf_{\substack{g(x)\le 0\\ h(x)=0}} p_0 f(x)$. $p_0 \ge 0$?

$$\Leftrightarrow \inf_{x\in\mathbb{R}^n} \left[\sup_{\substack{p\ge 0\\ q\in\mathbb{R}}} \left(p_0 f(x) + p\, g(x) + q\, h(x) \right) \right].$$

$\to$ penalization.

$$\begin{cases} +\infty & \text{if } g(x)>0 \text{ or } h(x)\neq 0 \\ p_0 f(x) & \text{if } g(x)\le 0 \text{ and } h(x)=0 \end{cases}$$

$\sim$. $\sup_{\substack{p\ge 0\\ q\in\mathbb{R}}} \inf_{x\in\mathbb{R}^n} \left(p_0 f(x) + p\, g(x) + q\, h(x) \right) =: L(x, p_0, p, q)$

$\to p_0 \nabla f(x^*) + p \nabla g(x^*) + q \nabla h(x^*) = 0$

Proof: ① $f_N(x) := f(x) + \|x-x^*\|^2 + \frac{N}{2}\left(\sum_{i\in I} \max(0, g_i(x))^2 + \sum_{j\in J} h_j^2(x) \right)$

$\ge f(x). \quad \forall x\in K.$

Moreover: $\boxed{f_N(x^*) = f(x^*)}$ and $f_N(x) > f(x) \geq f(x^*)$ $\forall x \neq x^*$, $x \in K$.

$\underset{\parallel}{f(x^*)}$ = $f_N(x^*)$

$\min_{x \in K} f_N(x)$ has a unique solution $x^* \in K$.

② Claim: $\exists\, \varepsilon_0 > 0$, s.t. $\forall\, \varepsilon \in [0, \varepsilon_0]$ $\exists\, N_\varepsilon > 0$

s.t. $f_{N_\varepsilon}(x) > f_{N_\varepsilon}(x^*)$ $\forall\, \|x - x^*\| = \varepsilon$.

$f_{N_\varepsilon}(x^*) = f(x^*)$, $x \notin K$.

$\min_{x \in \mathbb{R}^n} f_N(x)$

↳ the solution may not be x^*.

We consider unconstrained pb. $\min_{x : \|x - x^*\| < \varepsilon} f(x)$.

by last proposition in Part I.1.

Assume that x_ε^* be a solution, so that $\|x - x_\varepsilon^*\| < \varepsilon$.

$\Rightarrow$ By Euler condition. $\nabla f_{N_\varepsilon}(x_\varepsilon) = 0$

$g_i^+(x_\varepsilon^*)$.

$$\Longleftrightarrow 1 \cdot \nabla f(x_\varepsilon) + 2(x_\varepsilon - x^*) + N_\varepsilon \left(\sum_{i \in I} \max(0, g_i(x_\varepsilon^*)) \cdot \nabla g_i(x_\varepsilon^*) + \sum_{j \in J} h_j(x_\varepsilon^*) \cdot \nabla h_j(x_\varepsilon^*) \right) = 0$$

Let $\rho^\varepsilon := \sqrt{1 + N_\varepsilon^2 \sum_{i \in I} g_i^+(x_\varepsilon^*)^2 + N_\varepsilon^2 \sum_{j \in J} h_j(x_\varepsilon^*)^2} > 0.$

and set: $p_0^\varepsilon := \dfrac{1}{\rho^\varepsilon} \to 0$, $\quad p_i^\varepsilon := \dfrac{N_\varepsilon \cdot g_i^+(x_\varepsilon^*)}{\rho^\varepsilon} \geq 0$

$$q_j^\varepsilon := \frac{N_\varepsilon h_j(x_\varepsilon^*)}{\rho^\varepsilon} \in \mathbb{R}$$

$$\Rightarrow \boxed{(p_0^\varepsilon)^2 + \sum (p_i^\varepsilon)^2 + \sum (q_j^\varepsilon)^2 = 1} \Leftrightarrow \|(p_0^\varepsilon, p^\varepsilon, q^\varepsilon)\| = 1.$$

$$\Rightarrow p_0^\varepsilon \nabla f(x_\varepsilon^*) + 2p_0^\varepsilon (x_\varepsilon^* - x^*) + \sum_{i \in I} p_i^\varepsilon \nabla g_i(x_\varepsilon^*) + \sum q_j^\varepsilon \nabla h_j(x_\varepsilon^*) = 0$$

Let $\varepsilon \downarrow 0$ so that $\|x^* - x_\varepsilon^*\| \to 0 \Rightarrow x_\varepsilon^* \to x^*$.

By taking subsequence one has ε_n ... $\to 0$

$(p_0^{\varepsilon_n}, p^{\varepsilon_n}, q^{\varepsilon_n}) \to (p_0, p, q)$

s.t. $\|(p_0, p, q)\| = 1 \neq 0$

$$\Rightarrow \underline{p_0 \nabla f(x^*) + \sum_{i \in I} p_i \nabla g_i(x^*) + \sum q_j \nabla h_j(x^*) = 0}$$

Besides: If $g_i(x^*) < 0$, $\Rightarrow p_i^\varepsilon := N_\varepsilon g_i^+(x_\varepsilon) / \rho_\varepsilon = 0$ for $\varepsilon > 0$ small enough.

$\Rightarrow g_i(x_\varepsilon) < 0 \Rightarrow g_i^+(x_\varepsilon) = 0$

$\Rightarrow$ either $\underline{g_i(x^*) = 0}$. or. $\underline{p_i = 0}$.

$$\Rightarrow p_i g_i(x^*) = 0. \quad \forall i \in I.$$

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③ We finally prove the claim. fix $\varepsilon > 0$.

Assume that Claim is not true, then for all $N > 0$.

$\exists x_N$ s.t. $\|x_N - x^*\| = \varepsilon$. and $f_N(x_N) \leq f(x^*)$.

Then. $\{x_N\}_{N\geq 1}$ is a sequence in a compact set $\{x : \|x-x^*\| = \varepsilon\}$

$\Longrightarrow$ $x_{N_k} \to \bar{x}$ along a subsequence.

Besides. $\lim\limits_{N\to+\infty} \frac{N}{2} \left(\sum_{i\in I} g_i^+(x_N)^2 + \sum_{j\in J} h_j(x_N)^2 \right) < +\infty$

$\leq f_N(x_N) \leq f(x^*) < +\infty$

$\Longrightarrow$ $\lim\limits_{N\to+\infty} \sum_{i\in I} g_i^+(x_N)^2 + \sum h_j(x_N)^2 = 0$

$\Longrightarrow$ $\sum g_i^+(\bar{x})^2 + \sum h_j(\bar{x})^2 = 0 \Longrightarrow \bar{x} \in K$

($\bar{x}$ satisfies the constraints.)

$\Longrightarrow$ $\underline{f(\bar{x})} + \underline{\|\bar{x}-x^*\|^2} \leq \varliminf f_N(x_N) \leq \underline{f(x^*)}$

which is impossible since $\|\bar{x}-x^*\| = \varepsilon$.

and $f(x^*) \leq f(x)$, $\forall x \in K$. #

Recall. (P). $\inf_{x\in K} f(x)$, $K := \{x \in \mathbb{R}^n : \boxed{g_i(x) \le 0}, h_j(x) = 0\}$
$i=1,\cdots,m$, $j=1,\cdots,l$.

$$L(x, p_0, p, q) := p_0 f(x) + \sum_{i=1}^m p_i g_i(x) + \sum_{j=1}^l q_j h_j(x)$$

Thm. 1 If $\boxed{x^* \in K}$ is a solution to (P).
Then there exist $(p_0, p, q) \neq 0$. s.t. $\boxed{\sum_{i=1}^m p_i g_i(x^*) = 0}$
$p_0 \ge 0$, $p \in \mathbb{R}^m_+$, $q \in \mathbb{R}^l$.
and $\nabla_x L(x, p_0, p, q) = 0$

$p_i g_i(x^*) = 0$ $\forall i=1,\cdots,m$.

$p_i \ge 0$, $g_i(x^*) \le 0$ $\Rightarrow$ $p_i g_i(x^*) \le 0$

Remark: If $p_0 > 0$, then we can define. $L_0(x, p, q) = \frac{1}{p_0} L(x, p_0, p, q)$

$$= f(x) + \sum_{i=1}^m \frac{p_i}{p_0} g_i(x) + \sum \frac{q_j}{p_0} h_j(x)$$

Then: $\nabla_x L(x, p_0, p, q) = 0 \iff \nabla_x L_0(x, p, q) = 0$

$$\iff \nabla f(x) + \sum_{i=1}^m \frac{p_i}{\cancel{p_0}} \nabla g_i(x) + \sum_{j=1}^l \frac{q_j}{\cancel{p_0}} \nabla h_j(x)$$

$\to \lambda_i$, $\to \mu_j$.

2.3. Qualification condition.
(Mangasarian-Fromovitz)

(Mangasarian–Fromovitz)-

Proposition 2, Assume that $x^* \in K$, satisfies. $\boxed{\text{is a solution to } (P).}$

① . The family of vectors. $\{\nabla h_1(x), \dots, \nabla h_\ell(x)\}$ is ~~orthogonal~~ linear independent

i.e. $\sum_{j=1}^{\ell} \lambda_j \nabla h_j(x) = 0 \Rightarrow \lambda_j = 0, \ \forall j = 1, \dots, \ell.$

② there exist. a vector $v \in \mathbb{R}^n$, $v \neq 0$, s.t. $\langle \nabla h_j(x), v \rangle = 0, \ \forall j = 1, \dots, \ell$

and $\{ \langle \nabla g_i(x), v \rangle < 0$ or $g_i(x) < 0, \ \forall i = 1, \dots, m.$

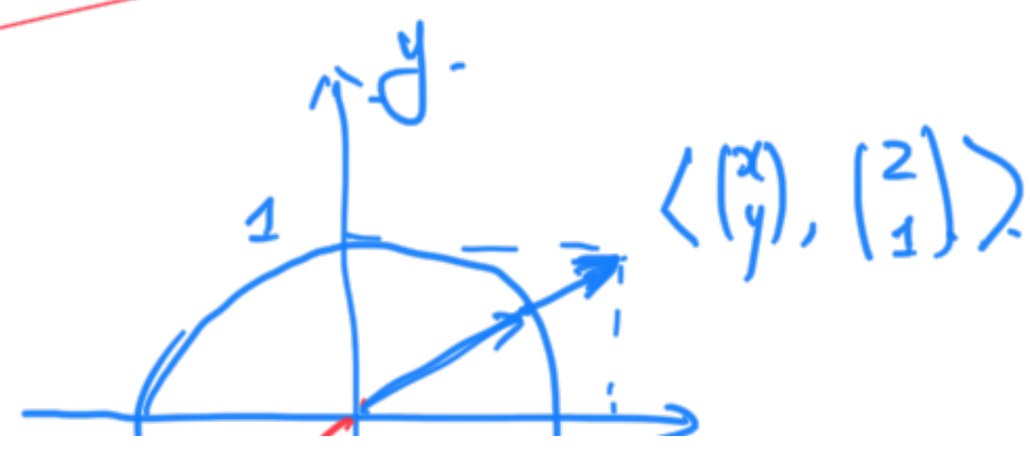
$\boxed{g_i(x) \le 0}$ → $g_i(x) = 0$, effective constraint.
→ $g_i(x) < 0$ non-effective constraint.

Then. $p_0 \neq 0$. in the Theorem of Kuhn–Tucker. $\left(p_0 \neq 0 \Rightarrow (p_0, p, q) \neq 0 \right)$

Consequently, there exist $\lambda \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^\ell$ s.t.

$$\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g(x^*) + \sum_{j=1}^{\ell} \mu_j \nabla h(x^*) = 0 \\ \sum_{i=1}^{m} \lambda_i g_i(x^*) = 0 \end{cases}$$

Example 1: $n = 2$. $\min. \ 2x + y$.



$x^2+y^2=1.$

$K=\{(x,y)\in\mathbb{R}^2 : h(x,y)=x^2+y^2-1=0\}$. is a compact.

- The optimal solution (x^*,y^*) exists.

$$\boxed{f(x,y)=2x+y \Rightarrow \nabla f(x,y)=\begin{pmatrix}2\\1\end{pmatrix}}$$

- $\nabla h(x^*,y^*)=\begin{pmatrix}2x^*\\2y^*\end{pmatrix}\neq 0$. since $x^{*2}+y^{*2}=1$.
  then (x^*,y^*) satisfies Qualification Condition ①
- Let $v=\begin{pmatrix}-y^*\\x^*\end{pmatrix}\neq 0$. then $\langle \nabla h(x^*,y^*), v\rangle$
  $=2\left\langle \begin{pmatrix}x^*\\y^*\end{pmatrix}, v\right\rangle=0$

So, (x^*,y^*) satisfies Qualification Condition ②.

- Then there exists $\mu\in\mathbb{R}$.

s.t. $\nabla f(x^*,y^*)+\mu\cdot\nabla h(x^*,y^*)=0 \iff \begin{pmatrix}2\\1\end{pmatrix}+\mu\begin{pmatrix}2x^*\\2y^*\end{pmatrix}=0.$

$$\Rightarrow \begin{cases}2+2\mu x^*=0\\1+2\mu y^*=0\\(x^*)^2+(y^*)^2=1.\end{cases}\Rightarrow\begin{cases}x^*=\frac{1}{\mu}=2y^*\\y^*=\frac{1}{2\mu}\\\underline{(x^*)^2+(y^*)^2=1.}\end{cases}\Rightarrow 4(y^*)^2+(y^*)^2=1$$

$$\Rightarrow\begin{cases}y^*=\pm\frac{\sqrt5}{5}\\x^*=2y^*.\end{cases}$$

$$\Rightarrow \begin{pmatrix}x^*\\y^*\end{pmatrix}=\begin{pmatrix}2\sqrt5/5\\\sqrt5/5\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}x^*\\y^*\end{pmatrix}=-\begin{pmatrix}2\sqrt5/5\\\sqrt5/5\end{pmatrix}.$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$f(x^*, y^*) = 2x^* + y^* = \sqrt{5} \qquad\qquad f(x^*, y^*) = -\sqrt{5}.$$

$\Rightarrow$ The optimal solution is $(x^*, y^*) = -(2\sqrt{5}/5, \sqrt{5}/5)$. #

Example. $\min\limits_{x^2+y^2 \le 1} xy$. $\Rightarrow$ $\underline{f(x,y) = x \cdot y}$.

$K = \{(x,y) : g(x,y) = \underline{x^2+y^2-1} \le 0\}$ is compact.

Then the optimal solution (x^*, y^*) exists.

$\underline{\nabla g(x^*, y^*)} = \underline{2\begin{pmatrix} x^* \\ y^* \end{pmatrix}}$, if $\begin{pmatrix} x^* \\ y^* \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

then for $v = -2\begin{pmatrix} x^* \\ y^* \end{pmatrix}$, one has $\langle \nabla g(x^*, y^*), v \rangle < 0$.

So. ① $\underline{\text{If } (x^*, y^*) = (0,0)}$, then it does not satisfies the Qualification Cond.

② If $(x^*, y^*) \neq (0,0)$, then it satisfies the Qualif. Cond.

For Case ②. there exists $\lambda \in \mathbb{R}_+$ s.t. $\nabla f(x^*, y^*) + \lambda g(x^*, y^*) = 0$ and $\underline{\lambda g(x^*, y^*) = 0}$

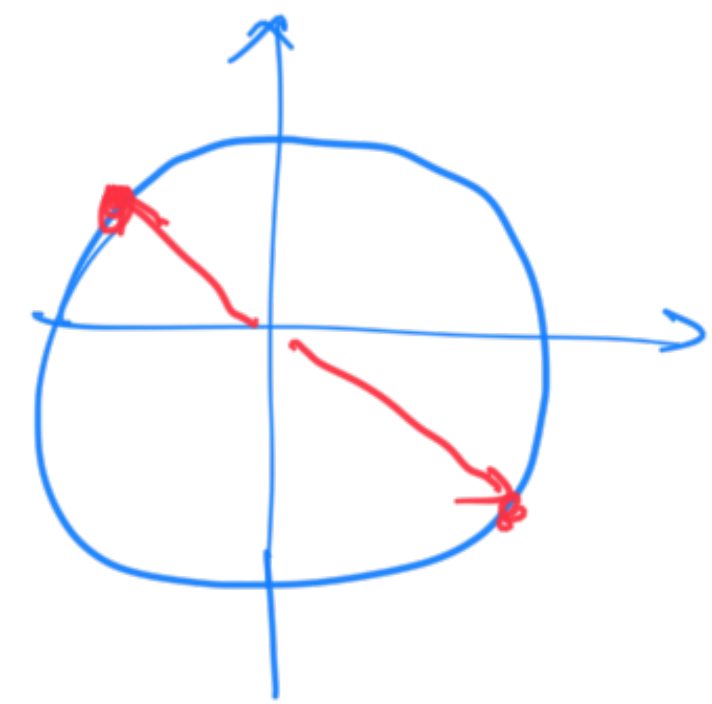
$$\Rightarrow \begin{cases} \begin{pmatrix} y^* \\ x^* \end{pmatrix} + 2\lambda \cdot \begin{pmatrix} x^* \\ y^* \end{pmatrix} = 0 \\ \lambda \cdot ((x^*)^2 + (y^*)^2 - 1) = 0 \end{cases} \Rightarrow \begin{cases} y^* = 2\lambda x^* = 4\lambda^2 y^* \\ x^* = 2\lambda y^* \\ \lambda\left((x^*)^2 + (y^*)^2 - 1\right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x^* = 0 \\ y^* = 0 \\ \lambda = 0 \end{cases} \text{ or } \begin{cases} x^* = \sqrt{2}/2 \\ y^* = \sqrt{2}/2 \\ \lambda = \frac{1}{2} \end{cases} \text{ or } \begin{cases} x^* = -\sqrt{2}/2 \\ y^* = -\sqrt{2}/2 \\ \lambda = \frac{1}{2} \end{cases} \text{ or } \begin{cases} x^* = \sqrt{2}/2 \\ y^* = -\sqrt{2}/2 \\ \lambda = -\frac{1}{2} \end{cases} \text{ or } \begin{cases} x^* = -\sqrt{2}/2 \\ y^* = \sqrt{2}/2 \\ \lambda = -\frac{1}{2} \end{cases}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$f(x^*, y^*) = 0 \qquad f(\cdot\cdot) = \frac{1}{2} \qquad f(\cdot\cdot) = \frac{1}{2} \qquad f(\cdot) = -\frac{1}{2} \qquad f(\cdot\cdot) = -\frac{1}{2}$$

$$\Rightarrow \begin{cases} x^* = \frac{\sqrt{2}}{2} \\ y^* = -\frac{\sqrt{2}}{2} \end{cases} \text{ or } \begin{cases} x^* = -\frac{\sqrt{2}}{2} \\ y^* = \frac{\sqrt{2}}{2} \end{cases}$$



Example 3: $\min \ x + 2y + 3z$

$$\underline{x^2 + y^2 + z^2 = 1}$$

$$x + y + z \le 0$$

$$\begin{cases} f(x, y, z) = x + 2y + 3z \\ g(x, y, z) = \underline{x + y + z \le 0} \end{cases}$$

1. Existence of (x^*, y^*, z^*). ✓

2. Qualif. Cond.

3. 1st order N.C.

$h(x,y,z) = x^2+y^2+z^2-1 = 0.$

Case ①. $(x^*, y^*, z^*) = 0$

Case ②: $(x^*, y^*, z^*) \neq 0.$

$$\nabla h(x,y,z) = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad \nabla g(x,y,z) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Case ②, $\nabla h(x^*, y^*, z^*) \neq 0. \Rightarrow \{\nabla h(x^*, y^*, z^*)\}$ is linear independent.

Next:, we aim to find a vector $v \in \mathbb{R}^3$ s.t.

$\langle \nabla h(x^*, y^*, z^*), v \rangle = 0.$ and $\begin{cases} \text{either } \langle \nabla g_i(x^*, y^*, z^*), v \rangle < 0 \\ \text{or. } g(x^*, y^*, z^*) < 0 \end{cases}$

If. $g(x^*, y^*, z^*) = x^* + y^* + z^* < 0$, then there exist. $v \perp \nabla h(x^*, y^*, z^*)$

so that $\langle \nabla h(\cdot\cdot), v \rangle = 0$

— Then Qualif. Cond. holds.

If $g(x^*, y^*, z^*) = x^* + y^* + z^* = 0.$ then for $v = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

one has: $\langle \nabla h(x^*, y^*, z^*), v \rangle = 2(x^* + y^* + z^*) = 0$

and $\langle \nabla g(\cdots), v \rangle = \langle \begin{pmatrix} 1 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ \end{pmatrix} \rangle$ ...

$$\left\langle \begin{pmatrix}1\\1\end{pmatrix}, -\begin{pmatrix}1\\1\end{pmatrix}\right\rangle = -3 < 0$$

- Then Qualif. Cond. holds.

By the last Proposition, there exist $\lambda \in \mathbb{R}_+$, $\mu \in \mathbb{R}$, s.t.

$$\begin{cases} \lambda g(x^*, y^*, z^*) = 0 \\ (\nabla f + \lambda \nabla g + \mu \nabla h)(x^*, y^*, z^*) = 0. \end{cases}$$

$$\Longrightarrow \begin{cases} \lambda(x^* + y^* + z^*) = 0. \\ \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} + 2\mu\begin{pmatrix}x^*\\y^*\\z^*\end{pmatrix} = 0. \end{cases} \Longrightarrow \begin{cases} \lambda(x^* + y^* + z^*) = 0 \\ 1 + \lambda + 2\mu x^* = 0 \\ 2 + \lambda + 2\mu y^* = 0 \\ 3 + \lambda + 2\mu z^* = 0. \\ (x^*)^2 + (y^*)^2 + (z^*)^2 = 1. \end{cases}$$

$\mu \neq 0$. and.

$$\begin{cases} x^* = -\dfrac{1+\lambda}{2\mu} \\ y^* = -\dfrac{2+\lambda}{2\mu} \\ z^* = -\dfrac{3+\lambda}{2\mu}. \end{cases} \Longrightarrow \begin{cases} \lambda\left(-\dfrac{1+\lambda + 2+\lambda + 3+\lambda}{2\mu}\right) = 0. \\ \dfrac{\underset{2}{(1+\lambda)^2} + \underset{4}{(2+\lambda)^2} + \underset{6.}{(3+\lambda)^2}}{4\mu^2} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda(6+3\lambda)=0 \\ 3\lambda^2+12\lambda+\underline{14 = 4\mu^2}. \end{cases} \Rightarrow \begin{cases} \lambda=0 \\ 4\mu^2=14 \end{cases} \text{ or } \begin{cases} \underline{\lambda=-2.} \\ \underline{4\mu^2=2.} \end{cases}$$

$\underline{12} - \underline{24} + \underline{14}$

$$\Rightarrow \begin{cases} \lambda=0 \\ \mu=\frac{\sqrt{14}}{2} \end{cases} \text{ or } \begin{cases} \lambda=0 \\ \mu=-\frac{\sqrt{14}}{2} \end{cases} \text{ or } \begin{cases} \lambda=-2 \\ \mu=\frac{\sqrt{2}}{2} \end{cases} \text{ or } \begin{cases} \lambda=-2 \\ \mu=-\frac{\sqrt{2}}{2}. \end{cases}$$

$$\Rightarrow \begin{cases} x^*=-\frac{1}{\sqrt{14}} \\ y^*=-\frac{2}{\sqrt{14}} \\ z^*=-\frac{3}{\sqrt{14}} \end{cases} \text{ or } \cancel{\begin{cases} x^*=1/\sqrt{14} \\ y^*=2/\sqrt{14} \\ z^*=3/\sqrt{14} \end{cases}} \text{ or } \begin{cases} x^*=\frac{1}{\sqrt{2}} \\ y^*=0 \\ z^*=-\frac{1}{\sqrt{2}} \end{cases} \text{ or. } \begin{cases} x^*=\frac{-1}{\sqrt{2}} \\ y^*=0 \\ z^*=\frac{1}{\sqrt{2}}. \end{cases}$$

$g(x^*,y^*,z^*) \le 0$

or. $x^*=y^*=z^*=0$

$f(x,y,z)= x+2y+3z.$

$f=-\sqrt{14}.$ or $f=\frac{1}{\sqrt{2}}-\frac{3}{\sqrt{2}}=\frac{-2}{\sqrt{2}}=-\sqrt{2}$ or $f=0$ or $f=0$

$$\Rightarrow \left\{ \begin{pmatrix} x^* \\ y^* \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Big/ \sqrt{14} \right. \quad \#$$

Proof of Proposition 2: We will use the contradiction arguments.

Assume that $p_0 = 0$ and $\sum_{i=1}^{m} \overset{\geq 0}{p_i}\, \overset{\leq 0}{g_i(x^*)} = 0$, $(p, q) \neq 0$

$$\cancel{p_0 \nabla f} + \sum_{i=1}^{m} p_i \nabla g_i(x^*) + \sum_{j=1}^{l} q_j \nabla h_j(x^*) = 0$$

Let $v \in \mathbb{R}^n$ be the vector in the Qualif. Cond.

$\Rightarrow \langle v, \sum_{i=1}^{m} p_i \nabla g_i(x^*) + \sum q_j \nabla h_j(x^*) \rangle = 0$

① $\langle v, \nabla h_j \rangle = 0$

$\Rightarrow \langle v, \sum_{i=1}^{m} p_i \nabla g_i(x^*) \rangle = 0$

② $p_i g_i(x^*) = 0 \quad \forall i$.

$\Leftrightarrow p_i = 0$ or $g_i(x^*) = 0 \quad \forall i$.

$p_i > 0$.

$\Rightarrow \sum_{i=1}^{m} p_i \langle v, \nabla g_i(x^*) \rangle = 0$

③ $\forall\, i = 1, \cdots, m$

either $\langle \nabla g_i(x^*), v \rangle < 0$

$\Rightarrow p_i \langle v, \nabla g_i(x^*) \rangle = 0 \quad \forall\, i = 1, \cdots m$.

$$\Rightarrow p_i = 0 \ , \ \forall i = 1, \cdots, m.$$

$$\Rightarrow \sum_{j=1}^{l} q_j \nabla h_j(x^*) = 0 \quad \Big\}$$

$\{\nabla h_j(x^*)\}$ is linear independent

$$\Rightarrow q_j = 0, \quad \forall j = 1, \cdots, l$$

$$\Rightarrow (p_0, \underline{p}, \underline{q}) = 0. \quad \text{Conditradition !} \quad \#$$

or $\underline{g_i(x^*) < 0}$

$\Rightarrow$ If $\underline{p_i = 0} \Rightarrow \underline{p_i \langle v, \nabla g_i(x^*)\rangle = 0}$

If $\underline{p_i > 0} \Rightarrow \underline{g_i(x^*) = 0} \Rightarrow \underline{\langle \nabla g_i(x^*), v\rangle < 0}$

$$\Rightarrow \underline{p_i \langle v, \nabla g_i(x^*)\rangle < 0}$$

Remark: $K = \{x \in \mathbb{R}^n : g_i(x) \le 0, \ h_j(x) = 0, \ \forall i, j\}$

The description of K by g and h is not unique.

$$K = \{x : \underline{x = 0}\} = \{x : \underline{x \le 0}, \ \underline{-x \le 0}\}$$

$h(x) = x = 0$ $\qquad g_1(x) = x \qquad g_2(x) = -x.$

"satisfies the Qualf. Cond."

$h(x) = ax+b$ convex.

$h(x)=0$

$\Leftrightarrow h(x)\le 0$, and $-h(x)\le 0$

3. Convex problem and the duality.

(P_c) : $\min_{x\in K} f(x)$, where $K = \{x\in\mathbb{R}^n : g_i(x)\le 0,\ i=1,\cdots,m\}$

$f, g_1, \cdots, g_m$ are all convex functions. $\in C^1$.

Proposition 3. Assume that the set $K^< := \{x\in\mathbb{R}^n : g_i(x) < 0,\ i=1,\cdots,m\} \ne \phi$.

Then $\forall x\in K$ satisfies the Qualf. Cond.

Proof: Let $\boxed{x_0\in K^<}$ be fixed. $\Rightarrow g_i(x_0)<0$.

Take any $x\in K$ $(x\ne x_0)$. and let $\boxed{V := x_0 - x \ne 0}$. $\Rightarrow g_i(x)\le 0$.

We will check that either $g_i(x)<0$ or $\langle V, \nabla g_i(x)\rangle < 0$, $\forall i=1,\cdots,m$

$g_i(x)=0$.

Indeed, as $g_i(x)$ is convex, if $g_i(x)=0$.

then $g_i(x_0) - g_i(x) \ge \langle \nabla g_i(x), x_0 - x\rangle$ (convexity of g_i.)

$$\Longleftrightarrow 0 > g_i(x_0) \geq \langle \nabla g_i(x), v \rangle.$$

$$\Longrightarrow \langle \nabla g_i(x), v \rangle < 0.$$

Thus the Qualif. Cond. holds. #