MMAT 5340 - Probability and Stochastic Analysis

II. Discrete time martingale

Stochastic process, filtration

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, a *stochastic process* is a family $(X_n)_{n\geq 0}$ of random variables indexed by time $n \geq 0$ (or $t_n,\ n \geq 0$).

A *filtration* is family $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ of sub-σ-field of $\mathcal{F}$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \geq 0$.

Example

Let $B = (B_n)_{n\geq 0}$ be some stochastic process, then the following definition of $\mathcal{F}_n$ provides a filtration $(\mathcal{F}_n)_{n\geq 0}$:

$$\mathcal{F}_n \ := \ \sigma(B_0, B_1, \cdots, B_n).$$

Example

In particular, let $B_0 = 0$, $B_n = \sum_{k=1}^n \xi_k$ where $(\xi_k)_{k\geq 1}$ is an i.i.d. sequence of random variables with distribution $\mathbb{P}[\xi_k = \pm 1] = \frac{1}{2}$. Then

$\mathcal{F}_0 = \{\emptyset, \Omega\},\ \mathcal{F}_1 = \mathcal{F}_0 \cup \{A, A^c\}$, *with* $A := \{\xi_1 = 1\},\ A^c = \{\xi_1 = -1\},\ \cdots$

Stochastic process, filtration

Let $X = (X_n)_{n\geq 0}$ be a stochastic process, $\mathbb{F} = (\mathcal{F}_n)_{n\geq 1}$ be a filtration.

- We say X is *adapted* to the filtration $\mathbb{F}$ if

$$X_n \in \mathcal{F}_n \text{ (i.e. } X_n \text{ is } \mathcal{F}_n\text{-measurable), for all } n \geq 0.$$

- We say X is *predictable* w.r.t. $\mathbb{F}$ if

$$X_n \in \mathcal{F}_{(n-1)\vee 0} \text{ for all } n \geq 0.$$

Remark: Let $\mathbb{F}$ be the filtration generated by the process B as in the above example. If X is $\mathbb{F}$-adapted, then $X_n \in \mathcal{F}_n = \sigma(B_0, \cdots, B_n)$ so that

$$X_n = g_n(B_0, \cdots, B_n), \text{ for some measurable function } g_n.$$

Similarly, if X is $\mathbb{F}$-predictable, then $X_{n+1} \in \mathcal{F}_n$ so that

$$X_{n+1} = g'_{n+1}(B_0, \cdots, B_n), \text{ for some measurable function } g'_{n+1}.$$

Stochastic process, filtration

Example

Let $(\xi_k)_{k\geq 1}$ be a sequence of i.i.d random variable, such that $\mathbb{P}\big[\xi_k = \pm 1\big] = \frac{1}{2}$. Then the process $X = (X_n)_{n\geq 0}$ defined as follows is called a random walk:

$$X_0 = 0, \quad X_n = \sum_{k=1}^{n} \xi_k.$$

Remark: Given a process X, let $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ be defined by $\mathcal{F}_n := \sigma(X_0, \cdots, X_n)$, we say $\mathbb{F}$ is the *natural filtration* generated by X.

In above examples, a stochastic process usually starts from time 0, but we can also consider stochastic process starting from some time t_k.

Martingale

Let $X = (X_n)_{n \geq 0}$ be a stochastic process, $\mathbb{F} = (\mathcal{F}_n)_{n \geq 1}$ be a filtration.

- We say X is a *martingale (w.r.t. $\mathbb{F}$)* if X is $\mathbb{F}$-adapted, each random variable X_n is integrable, and

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n.$$

- We say X is a *sub-martingale (w.r.t. $\mathbb{F}$)* if X is $\mathbb{F}$-adapted, each random variable X_n is integrable, and

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n.$$

- We say X is a *super-martingale (w.r.t. $\mathbb{F}$)* if X is $\mathbb{F}$-adapted, each random variable X_n is integrable, and

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n.$$

Remark: A martingale X (w.r.t. to some filtration $\mathbb{F}$) is a sub-martingale, and at the same time a super-martingale.

Martingale

Example

Recall that the random walk $X = (X_n)_{n\geq 0}$ *is defined as follows:*

$$X_0 = 0, \quad X_n = \sum_{k=1}^{n} \xi_k,$$

where $(\xi_k)_{k\geq 1}$ *be a sequence of i.i.d. of random variable such that* $\mathbb{P}[\xi = \pm 1] = \frac{1}{2}$.
Then

- X *is a martingale;*
- $(X_n^2)_{n\geq 0}$ *is a sub-martingale;*
- $(X_n^2 - n)_{n\geq 0}$ *is a martingale.*

Martingale

Example

Let $(Z_k)_{k\geq 1}$ *be a sequence of random variable such that* $Z_k \sim N(0,1)$*, and* $\sigma \in \mathbb{R}$*,* $X_0 \in \mathbb{R}$ *be real constants. Let* $\mathcal{F}_n := \sigma(Z_1, \cdots, Z_n)$*, and*

$$X_n \; := \; X_0 \exp\Big(\sigma \sum_{k=1}^{n} Z_k - \frac{1}{2} n \sigma^2\Big).$$

Then $(X_n)_{n\geq 1}$ *is a martingale (w.r.t.* $\mathbb{F}$*).*

Example

Let $\mathbb{F} = (\mathcal{F}_n)_{n\geq 1}$ *be a filtration,* Z *be an integrable random variable, and*

$$X_n \; := \; \mathbb{E}\big[Z \big| \mathcal{F}_n\big].$$

Then $(X_n)_{n\geq 1}$ *is a martingale (w.r.t.* $\mathbb{F}$*).*

Martingale

Lemma

Let $\mathbb{F}$ be a filtration, and X be a martingale w.r.t. $\mathbb{F}$. Let $\mathbb{F}^X$ denote the natural filtration generated by X. Then X is also a martingale w.r.t. $\mathbb{F}^X$.

Remark: We notice that a martingale X is associated to some filtration $\mathbb{F}$. However, when the filtration is not specified, we say X is a martingale means that X is a martingale w.r.t. the natural filtration generated by X. In this case, we write

$$\mathbb{E}\big[X_{n+1}\big|X_0, \cdots, X_n\big] = X_n, \quad \text{for all } n \geq 0.$$

Martingale

Lemma

Let X be a martingale w.r.t. the filtration $\mathbb{F}$, then

$$\mathbb{E}\big[X_m \big| \mathcal{F}_n\big] \;=\; X_n, \text{ for all } m \geq n \geq 0.$$

Moreover,

$$\mathbb{E}\big[X_n\big] = \mathbb{E}\big[X_0\big], \text{ for all } n \geq 0.$$

Stopping time

Definition: Let $\mathbb{F}$ be a filtration, a *stopping time* w.r.t. $\mathbb{F}$ is a random variable $\tau : \Omega \longrightarrow \{0, 1, \cdots\} \cup \{\infty\}$ such that

$$\{\tau \leq n\} \in \mathcal{F}_n, \quad \text{for all } n \geq 0. \tag{1}$$

Remark: In place of (1), it is equivalent to define the stopping time by the property:

$$\{\tau = n\} \in \mathcal{F}_n, \quad \text{for all } n \geq 0.$$

Lemma

Let X be a stochastic process adapted to the filtration $\mathbb{F}$, and B be a Borel set in $\mathbb{R}$. Then the hitting time τ defined below is a stopping w.r.t. $\mathbb{F}$:

$$\tau \;:=\; \inf\{n \geq 0 \; : X_n \in B\}.$$

Remark: In above, we use the convention $\inf \emptyset = +\infty$.

Doob's optional stopping theorem

Given a stochastic process X and a stopping time τ w.r.t. some filtration $\mathbb{F}$.

$$X_{\tau\wedge n}(\omega) \ := \begin{cases} X_n(\omega) & \text{if } \tau(\omega) \geq n, \\ X_{\tau(\omega)}(\omega) & \text{if } \tau(\omega) < n. \end{cases}$$

Theorem

Let $\mathbb{F}$ be fixed filtration, X be a $\mathbb{F}$-martingale, and τ be a $\mathbb{F}$-stopping time. Then the process $(X_{\tau\wedge n})_{n\geq 0}$ is still a $\mathbb{F}$-martingale.

Remark: When X is martingale and τ is a stopping w.r.t. the same filtration, it follows that

$$\mathbb{E}\big[X_{\tau\wedge n}\big] = \mathbb{E}\big[X_0\big].$$

Doob's optional stopping theorem

Theorem

Let $\mathbb{F}$ be fixed filtration, X be a $\mathbb{F}$-martingale, and τ be a $\mathbb{F}$-stopping time. Assume that τ is bounded by some constant $m \geq 0$, or the process $(X_{\tau\wedge n})_{n\geq 0}$ is uniformly bounded. Then

$$\mathbb{E}\big[X_\tau\big] = \mathbb{E}\big[X_0\big].$$

Doob's optional stopping theorem

Example

Let $(\xi_k)_{k\ge 1}$ *be a sequence of i.i.d. random variables,* $x \in \mathbb{N}$ *be a positive integer, and*

$$X_n \;:=\; x + \sum_{k=1}^{n} \xi_k.$$

Let us define

$$\tau \;:=\; \inf\big\{n \ge 0 \;: X_n \le 0 \text{ or } X_n \ge N\big\}.$$

Compute the value of $\mathbb{E}\big[X_\tau\big]$*. Compute the probability* $\mathbb{P}\big[X_\tau = 0\big]$*.*

Convergence of martingale

Theorem

Let X *be a submartingale such that* $\sup_{n\geq 0}\mathbb{E}[|X_n|]<\infty$. *Then*

$$\lim_{n\to\infty} X_n = X, \text{ for some r.v. } X \in L^1.$$

Convergence of martingale

Theorem

Let X be a martingale such that $\sup_{n\geq 0}\mathbb{E}[|X_n|^2]<\infty$. Then

$$\lim_{n\to\infty} X_n = X, \text{ for some r.v. } X\in L^2.$$

Application: Law of large number

Theorem (Law of large number)

Let $(\xi_k)_{k\geq 1}$ *be a sequence of i.i.d. random variables, such that* $\mathbb{E}[|\xi_i|] < \infty$*. Then*

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k \longrightarrow \mathbb{E}[X_1], \text{ a.s.}$$

We will use the theorem of convergence of martingale to prove the above theorem.