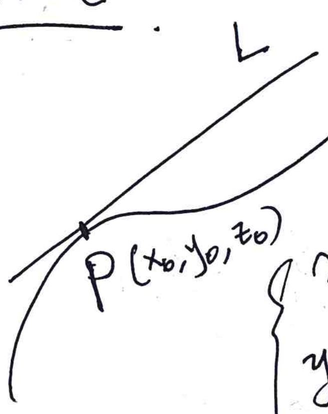
Curve



$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$ differentiable at t_0

parameter equation for curve

$x'^2(t_0) + y'^2(t_0) + z'^2(t_0) \neq 0$.

$x_0 = x(t_0)$

$y_0 = y(t_0)$

$z_0 = z(t_0)$

tangent line: $\dfrac{x - x_0}{x'(t_0)} = \dfrac{y - y_0}{y'(t_0)} = \dfrac{z - z_0}{z'(t_0)}$

parameter equation for surface

Surface: $$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$ diff at (u_0, v_0)

and ran $\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{pmatrix}_{(u_0, v_0)} = 2.$

set $A = \cancel{k}\ \dfrac{\partial(y,z)}{\partial(u,v)}\Big|_P$, $B = \dfrac{\partial(z,x)}{\partial(u,v)}\Big|_P$

$$C = \frac{\partial(x,y)}{\partial(u,v)}\Big|_P \quad \left(\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \right.$$

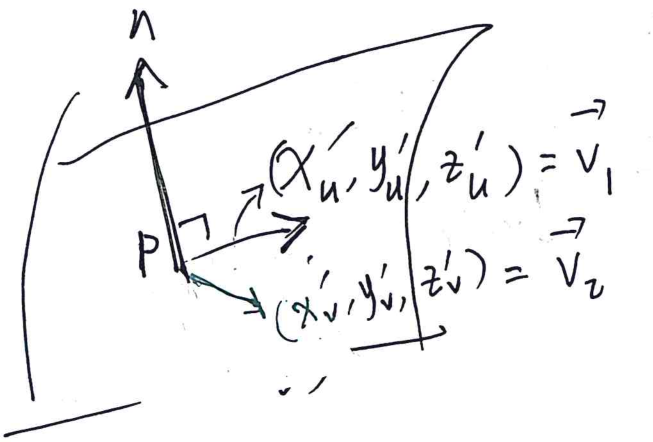
$\Rightarrow$ normal vector at P.

$$\vec{n} = \begin{vmatrix} i & j & k \\ x_u' & y_u' & z_u' \\ x_v' & y_v' & z_v' \end{vmatrix} = (A, B, C) \left(= \vec{v}_1 \times \vec{v}_2 \right)$$

$\Rightarrow$ tangent plane of S passing through P

$$: \ A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

normal line: $\dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}$



① $z = f(x, y)$. dift at (x_0, y_0)

$\downarrow u$ $\downarrow v$.

tangent plane at $P(x_0, y_0, z_0)$:

$$z - z_0 = f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0)$$

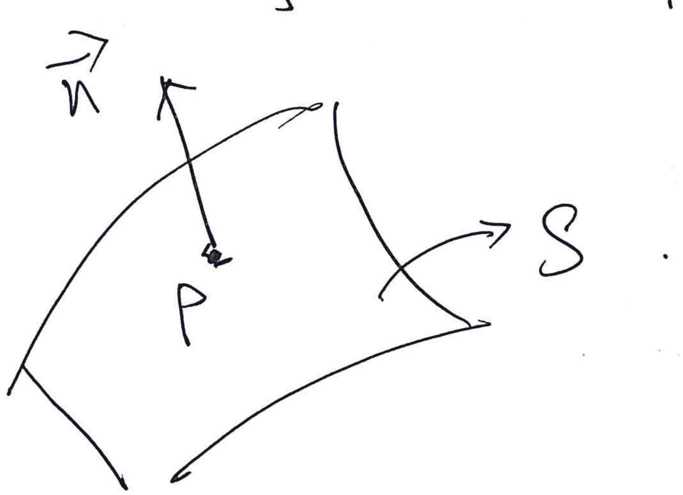
② $F(x, y, z) = 0$. $P(x_0, y_0, z_0)$.

F dif. at P. ${F_x'}^2 + {F_y'}^2 + {F_z'}^2 \neq 0$ at P

$\Rightarrow$ tangent plane passing P:

$$F_x'(p)(x - x_0) + F_y'(p)(y - y_0) + F_z'(p)(z - z_0) = 0.$$

$(F_x'(p), F_y'(p), F_z'(p))$ is the normal vector of S at P.



ex: S: $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$.

for any P on S, there is a tangent plane passing through P. intersect with x, y, z axis. $(\bar{x},0,0)$, $(0,\bar{y},0)$, $(0,0,\bar{z})$ three points. prove that: for any P. the value $\bar{x}+\bar{y}+\bar{z}$ doesn't change.

proof: normal vector at (x_0, y_0, z_0) is $\left(\frac{1}{2\sqrt{x_0}}, \frac{1}{2\sqrt{y_0}}, \frac{1}{2\sqrt{z_0}}\right)$.

$\Rightarrow$ tangent plane: $\frac{x-x_0}{\sqrt{x_0}}+\frac{y-y_0}{\sqrt{y_0}}+\frac{z-z_0}{\sqrt{z_0}}=0$

$$\Rightarrow \bar{x}=x_0+\sqrt{x_0}\left(\sqrt{y_0}+\sqrt{z_0}\right)$$

$$\bar{y}=y_0+\sqrt{y_0}\left(\sqrt{x_0}+\sqrt{z_0}\right)$$

$$\bar{z}=z+\sqrt{z_0}\left(\sqrt{y_0}+\sqrt{x_0}\right).$$

$$\Rightarrow \bar{x}+\bar{y}+\bar{z}=a.$$

ex : find the tangent plane of

$S: x^2+2y^2+3z^2=21$ s.t it's paralleled

to $x+4y+6z=0$.

sol: normal vector $\begin{matrix} F_x' & F_y' & F_z' \\ (x, & 2y, & 3z) \end{matrix}$

by parallel. $\Rightarrow \frac{x}{1} = \frac{2y}{4} = \frac{3z}{6} = \lambda$

$\Rightarrow \lambda = \pm 1$

$\Rightarrow$ two tangent point.

$(x,y,z) = (\pm 1, \pm 2, \pm 2)$

$\Rightarrow (x \mp 1) + 4(y \mp 2) + 6(z \mp 2) = 0$.

chain rule :

ex : check that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

after the polar coodinate.

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y \partial x}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

$$= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta \quad (2)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r\left(-\frac{\partial^2 u}{\partial x^2} r\sin\theta + \frac{\partial^2 u}{\partial x \partial y} r\cos\theta\right)\sin\theta - r\frac{\partial u}{\partial x}\cos\theta$$

$$+ r\left(-\frac{\partial^2 u}{\partial x \partial y} r\sin\theta + r\frac{\partial^2 u}{\partial y^2}\cos\theta\right)\cos\theta - r\frac{\partial u}{\partial y}\sin\theta$$

$$= r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 u}{\partial x \partial y} r^2\cos\theta\sin\theta + r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta$$

$$- r\frac{\partial u}{\partial x}\cos\theta - r\frac{\partial u}{\partial y}\sin\theta$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - \frac{2\,\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta \quad (1)$$

$$\underbrace{-\frac{1}{r}\frac{\partial u}{\partial y}\sin\theta - \frac{1}{r}\frac{\partial u}{\partial x}\cos\theta}_{= \frac{1}{r}\frac{\partial u}{\partial r}}$$

$$(1) + (2) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}. \quad \text{namely,}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$