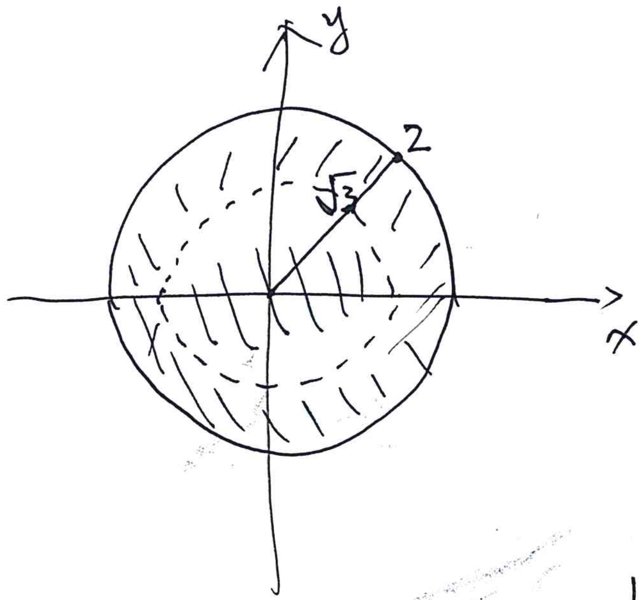
Domain of a function

ex: 1. $f(x,y) = \dfrac{1}{\ln(4-x^2-y^2)}$

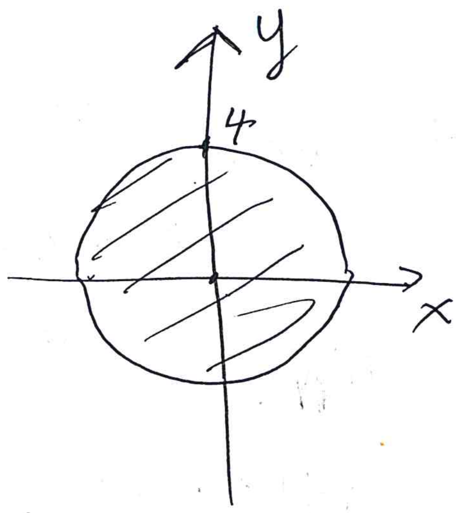
$$\begin{cases} 4-x^2-y^2 > 0 \\ \ln(4-x^2-y^2) \neq 0. \end{cases}$$

$$\Rightarrow \begin{cases} 4-x^2-y^2 > 0 \\ 4-x^2-y^2 \neq 1. \end{cases} \Rightarrow \quad x^2+y^2 < 4.$$

but $x^2+y^2 \neq 3$.



2. $f(x,y) = \dfrac{1}{\sqrt{16-x^2-y^2}}$



$16-x^2-y^2 > 0 \Rightarrow x^2+y^2 < 16$

limit.

1. $\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$.

continuous at point $(0,4)$.

substituting $(0,4)$ into the function $\frac{x}{\sqrt{y}}$ is ok

$\Rightarrow = \frac{0}{\sqrt{4}} = 0$.

2. $\lim_{(x,y)\to(0,0)} \cos \frac{x^2+y^3}{x+y+1}$

same reason with 1

$= \cos \frac{0+0}{0+0+1} = \cos 0 = 1$

3. $\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy}$.

$(x,y)\to(0,0) \Rightarrow xy \to 0$.

$\sin^2 \frac{xy}{2} = \frac{1-\cos xy}{2}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2\sin^2\frac{xy}{2}}{xy}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin^2\frac{xy}{2}}{\frac{xy}{2}}$$

set $z = \frac{xy}{2}$

$$= \lim_{z\to 0} \frac{\sin^2 z}{z} = \lim_{z\to 0}\left(\frac{\sin z}{z}\right)^2 \cdot z = 0.$$

4. ~~(x+y~~ $\lim_{(x,y)\to(0,0)} \underbrace{(x+y)\ln(x^2+y^2)}_{f(x,y)}$

polar coodinate method.

set. $x = r\cos\theta$, $y = r\sin\theta$.

$$\Rightarrow \left|(x+y)\ln(x^2+y^2)\right| = \left|r(\cos\theta+\sin\theta)\ln r^2\right|$$

$$\leq \left|4\underbrace{r\ln r}_{\to 0,\ r\to 0}\right| \Rightarrow f(x,y)\to 0 \quad \left((x,y)\to(0,0)\right)$$

5. $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2}$$

set $x = r\cos\theta$, $y = r\sin\theta$.

$$\Rightarrow 0 \le \left|\frac{\sin(x^3+y^3)}{x^2+y^2}\right| \le \left|\frac{x^3+y^3}{x^2+y^2}\right| = \left|r(\cos^3\theta+\sin^3\theta)\right|$$

$$\downarrow$$

$$0 \qquad\qquad \le 2r \to 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2} = 0.$$

6. $$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{x^2+y^2}{x^4+y^4} \quad \cancel{(f(x,y))}$$

$$0 < f(x,y) = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \le \frac{1}{x^2}+\frac{1}{y^2}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \ {\scriptstyle x\to\infty,\ y\to\infty}$$

$$0 \qquad\qquad\qquad\qquad 0$$

$$\Downarrow$$

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{x^2+y^2}{x^4+y^4} = 0.$$

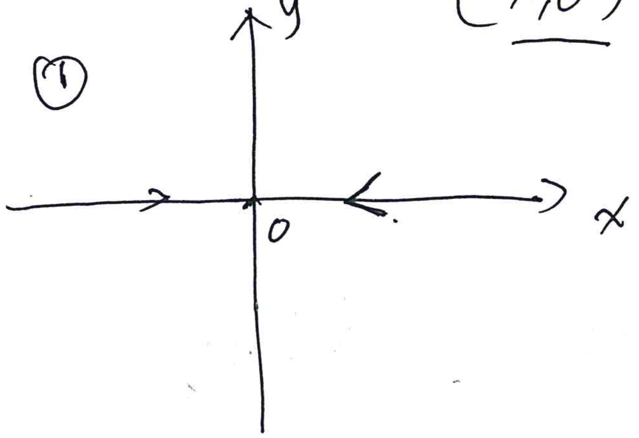
7. does the limit $\lim\limits_{(x,y)\to(0,0)} \frac{\ln(1+xy)}{x+\tan y}$ exist?

No!

answer: see ~~th~~ two different approximate direction.

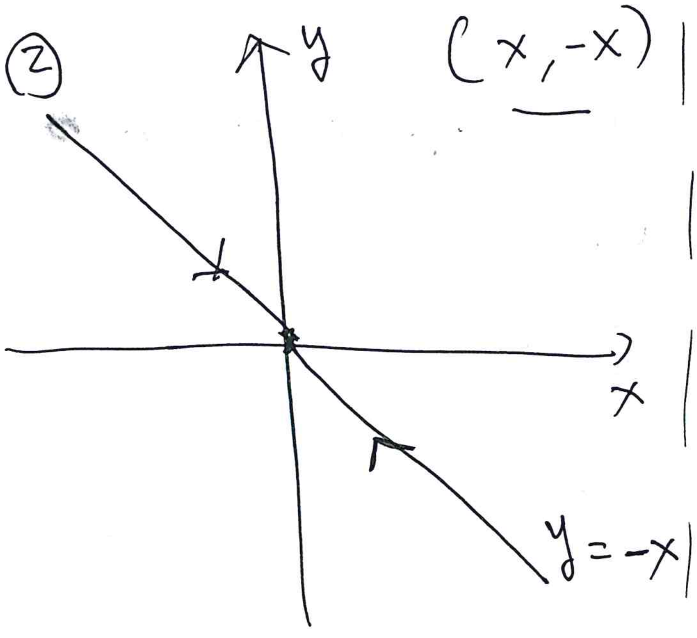
set $f(x,y) = \frac{\ln(1+xy)}{x+\tan y}$

① $(x,0)$



$\Rightarrow f(x,0) = 0$

② $(x,-x)$



$f(x,-x) = \frac{\ln(1-x^2)}{x-\tan x}$

$= \frac{-x^2}{x-\tan x} \cdot \ln(1-x^2)^{-\frac{1}{x^2}}$

$\frac{-x^2}{x-\tan x} \to +\infty$, $\ln(1-x^2)^{-\frac{1}{x^2}} \to 1$

$\Downarrow$

$\lim\limits_{x\to 0} f(x,-x) = \infty \neq 0$

$\Downarrow$

limit does not exist.

reason on the back.

$$\lim_{x \to 0} \frac{-x^2}{x - \tan x}$$

first see. $\lim_{x \to 0} \frac{x - \tan x}{-x^2}$

by L'Hospital $= \lim_{x \to 0} \frac{1 - \sec^2 x}{-2x}$ $\left(\sec x = \frac{1}{\cos x}\right)$

L'Hospital a second time

$$= \lim_{x \to 0} \frac{0 - 2\sec x \cdot \sec x \tan x}{-2}$$

$(\because (\sec x)' = \sec x \tan x)$ $\quad \| \quad$ $\tan x \to 0 \ (x \to 0)$

$0.$

$$\Rightarrow \lim_{x \to 0} \frac{-x^2}{x - \tan x} = \infty$$