MATH 2010A/B Advanced Calculus I
(2014-2015, First Term)
Homework 10
Suggested Solution

15. $f(x,y) = 3x^2 + 6xy + 2y^3 + 12x - 24y$.
$f_x = 6x + 6y + 12$; $f_y = 6x + 6y^2 - 24$.
$f_{xx} = 6$; $f_{xy} = 6$; $f_{yy} = 12y$.
Solving

$$\begin{cases} 6x + 6y + 12 &= 0 \\ 6x + 6y^2 - 24 &= 0 \end{cases}$$

We have $(x,y) = (0,-2)$, $(-5,3)$.
At $(x,y) = (0,-2)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = 180 > 0$. Thus $(0,-2)$ is a saddle point.
At $(x,y) = (-5,3)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = -180 < 0$ and $f_{xx} = 6 > 0$. Thus $(-5,3)$ is a local minimum point with value $f(-5,3) = -93$. This is the only local minimum point, thus it is also the global minimum point.

20. $f(x,y) = 2x^3 + y^3 - 3x^2 - 12x - 3y$.
$f_x = 6x^2 - 6x - 12$; $f_y = 3y^2 - 3$.
$f_{xx} = 12x - 6$; $f_{xy} = 0$; $f_{yy} = 6y$.
Solving

$$\begin{cases} 6x^2 - 6x - 12 &= 0 \\ 3y^2 - 3 &= 0 \end{cases}$$

We have $(x,y) = (2,1)$, $(-1,1)$, $(2,-1)$, $(-1,-1)$.
At $(x,y) = (2,1)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = -108 < 0$ and $f_{xx} = 18 > 0$. Thus $(2,1)$ is a local minimum point with value $f(2,1) = -22$.
At $(x,y) = (-1,1)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = 108 > 0$. Thus $(-1,1)$ is a saddle point.
At $(x,y) = (2,-1)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = 108 > 0$. Thus $(2,-1)$ is a saddle point.
At $(x,y) = (-1,-1)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = -108 < 0$ and $f_{xx} = -18 < 0$. Thus $(-1,-1)$ is a local maximum point with value $f(-1,-1) = 9$.
In conclusion, $(2,1)$ is the global minimum point and $(-1,-1)$ is the global maximum point.

23. $f(x,y) = x^4 + y^4$.
$f_x = 4x^3$; $f_y = 4y^3$.
$f_{xx} = 12x^2$; $f_{xy} = 0$; $f_{yy} = 12y^2$.
Then at $(x,y) = (0,0)$, $\triangle = f_{xx}f_{yy} - f_{xy}^2 = 0$.
When $|x|, |y| \to \infty$, $f(x,y) \to \infty$. Thus f is open upwards and $(0,0)$ is the global minimum point with value $f(0,0) = 0$.

25. $f(x,y) = e^{-x^4-y^4}$.
$f_x = -4x^3e^{-x^4-y^4}$; $f_y = -4y^3e^{-x^4-y^4}$.
$f_{xx} = 4x^2e^{-x^4-y^4}(4x^4-3)$; $f_{xy} = 16x^3y^3exp(-x^4-y^4)$; $f_{yy} = 4y^2e^{-x^4-y^4}(4y^4-3)$.
Then at $(x,y) = (0,0)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = 0$.
Note that $e^{x^4+y^4} \geq 1$ for any (x,y). Thus $e^{-x^4-y^4} \leq 1 = f(0,0)$ for any (x,y). Therefore, $(0,0)$ is the global maximum point with value 1.

31. $f(x,y) = \sin\frac{\pi x}{2}\sin\frac{\pi y}{2}$.
$f_x = \frac{\pi}{2}\cos\frac{\pi x}{2}\sin\frac{\pi y}{2}$; $f_y = \frac{\pi}{2}\sin\frac{\pi x}{2}\cos\frac{\pi y}{2}$.
$f_{xx} = -\frac{\pi^2}{4}\sin\frac{\pi x}{2}\sin\frac{\pi y}{2}$; $f_{xy} = \frac{\pi^2}{4}\cos\frac{\pi x}{2}\cos\frac{\pi y}{2}$; $f_{yy} = -\frac{\pi^2}{4}\sin\frac{\pi x}{2}\sin\frac{\pi y}{2}$.
Solving

$$\begin{cases} \frac{\pi}{2}\cos\frac{\pi x}{2}\sin\frac{\pi y}{2} &= 0 \\ \frac{\pi}{2}\sin\frac{\pi x}{2}\cos\frac{\pi y}{2} &= 0 \end{cases}$$

$(x,y) = (4m \pm 1, 4n \pm 1),\ m,n \in \mathbb{Z}$ or $(x,y) = (2m,2n),\ m,n \in \mathbb{Z}$.

When $(x,y) = (4m \pm 1, 4n \pm 1)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = -\frac{\pi^4}{16} < 0$.

Case 1: $(x,y) = (4m+1, 4n+1)$, then $f_{xx} = -\frac{\pi^2}{2} < 0$. Thus local maximum point with value $f(4m+1, 4n+1) = 1$.

Case 2: $(x,y) = (4m-1, 4n-1)$, then $f_{xx} = -\frac{\pi^2}{2} < 0$. Thus local maximum point with value $f(4m-1, 4n-1) = 1$.

Case 3: $(x,y) = (4m+1, 4n-1)$, then $f_{xx} = \frac{\pi^2}{2} > 0$ Thus local minimum point with value $f(4m+1, 4n-1) = -1$.

Case 4: $(x,y) = (4m-1, 4n+1)$, then $f_{xx} = \frac{\pi^2}{2} > 0$ Thus local minimum point with value $f(4m+1, 4n+1) = -1$.

When $(x,y) = (2m,2n)$, $\triangle = f_{xy}^2 - f_{xx}f_{yy} = \frac{\pi^4}{4} > 0$. Thus saddle point.

Exercises 14.9

7. $f(x,y) = \sin(x^2+y^2)$.
$f_x = 2x\cos(x^2+y^2)$; $f_y = 2y\cos(x^2+y^2)$.
$f_{xx} = 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)$; $f_{xy} = -4xy\sin(x^2+y^2)$; $f_{yy} = 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)$.
The quadratic approximation at the origin is

$$\begin{aligned} f(x,y) &= f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left(x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right) \\ &= 0 + x\cdot 0 + y \cdot 0 + \frac{1}{2}\left(x^2\cdot 2 + 2xy\cdot 0 + y^2 \cdot 2\right) \\ &= x^2 + y^2 \end{aligned}$$

9. $f(x,y) = \frac{1}{1-x-y}$.
$f_x = \frac{1}{(1-x-y)^2}$; $f_y = \frac{1}{(1-x-y)^2}$.
$f_{xx} = \frac{2}{(1-x-y)^3}$; $f_{xy} = \frac{2}{(1-x-y)^3}$; $f_{yy} = \frac{2}{(1-x-y)^3}$.

The quadratic approximation at the origin is

$$
\begin{aligned}
f(x,y) &= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left(x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right) \\
&= 1 + x + y + \left(x^2 + 2xy + y^2\right) \\
&= 1 + (x+y) + (x+y)^2
\end{aligned}
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