In Problems 1 through 12, find every point on the given surface $z = f(x, y)$ at which the tangent plane is horizontal.

1. $z = x - 3y + 5$

2. $z = 4 - x^2 - y^2$

3. $z = xy + 5$

4. $z = x^2 + y^2 + 2x$

5. $z = x^2 + y^2 - 6x + 2y + 5$

6. $z = 10 + 8x - 6y - x^2 - y^2$

7. $z = x^2 + 4x + y^3$

8. $z = x^4 + y^3 - 3y$

9. $z = 3x^2 + 12x + 4y^3 - 6y^2 + 5$ (Fig. 12.5.13)

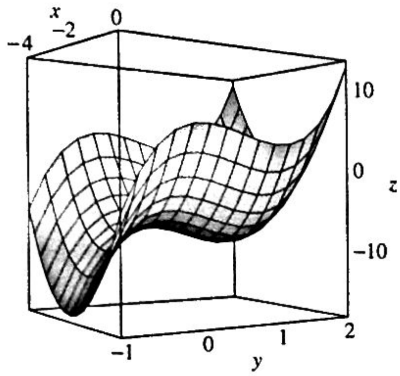

FIGURE 12.5.13 The surface of Problem 9.

10. $z = \dfrac{1}{1 - 2x + 2y + x^2 + y^2}$

11. $z = (2x^2 + 3y^2)\exp(-x^2 - y^2)$ (Fig. 12.5.14)

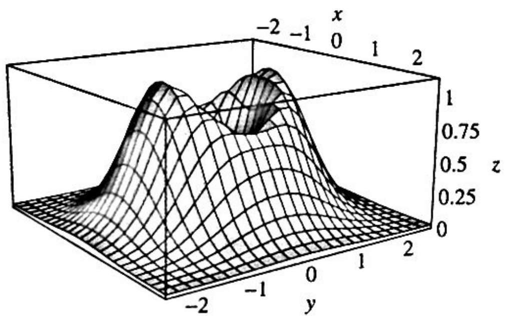

FIGURE 12.5.14 The surface of Problem 11.

12. $z = 2xy\exp\left(-\frac{1}{8}(4x^2 + y^2)\right)$ (Fig. 12.5.15)

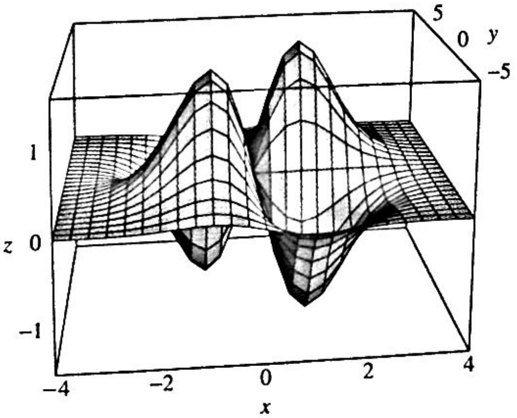

FIGURE 12.5.15 The surface of Problem 12.

Each of the surfaces defined in Problems 13 through 22 either opens downward and has a highest point, or opens upward and has a lowest point. Find this highest or lowest point on the surface $z = f(x, y)$.

13. $z = x^2 - 2x + y^2 - 2y + 3$

14. $z = 6x - 8y - x^2 - y^2$

15. $z = 2x - x^2 + 2y^2 - y^4$

16. $z = 4xy - x^4 - y^4$

17. $z = 3x^4 - 4x^3 - 12x^2 + 2y^2 - 12y$

18. $z = 3x^4 + 4x^3 + 6y^4 - 16y^3 + 12y^2$

19. $z = 2x^2 + 8xy + y^4$

20. $z = \dfrac{1}{10 - 2x - 4y + x^2 + y^4}$

21. $z = \exp(2x - 4y - x^2 - y^2)$

22. $z = (1 + x^2)\exp(-x^2 - y^2)$

In Problems 23 through 28, find the maximum and minimum values attained by the given function $f(x, y)$ on the given plane region R.

23. $f(x, y) = x + 2y$; R is the square with vertices at $(\pm 1, \pm 1)$.

24. $f(x, y) = x^2 + y^2 - x$; R is the square of Problem 23.

25. $f(x, y) = x^2 + y^2 - 2x$; R is the triangular region with vertices at $(0, 0)$, $(2, 0)$, and $(0, 2)$.

26. $f(x, y) = x^2 + y^2 - x - y$; R is the region of Problem 25.

27. $f(x, y) = 2xy$; R is the circular disk $x^2 + y^2 \leqq 1$.

28. $f(x, y) = xy^2$; R is the circular disk $x^2 + y^2 \leqq 3$.

In Problems 29 through 34, the equation of a plane or surface is given. Find the first-octant point $P(x, y, z)$ on the surface closest to the given fixed point $Q(x_0, y_0, z_0)$. [Suggestion: Minimize the squared distance $|PQ|^2$ as a function of x and y.]

29. The plane $12x + 4y + 3z = 169$ and the fixed point $Q(0, 0, 0)$

30. The plane $2x + 2y + z = 27$ and the fixed point $Q(9, 9, 9)$

31. The plane $2x + 3y + z = 49$ and the fixed point $Q(7, -7, 0)$

32. The surface $xyz = 8$ and the fixed point $Q(0, 0, 0)$

33. The surface $x^2y^2z = 4$ and the fixed point $Q(0, 0, 0)$

34. The surface $x^4y^8z^2 = 8$ and the fixed point $Q(0, 0, 0)$

35. Find the maximum possible product of three positive numbers whose sum is 120.

36. Find the maximum possible volume of a rectangular box if the sum of the lengths of its 12 edges is 6 meters.

37. Find the dimensions of the box with volume 1000 in.^{3} that has minimal total surface area.

38. Find the dimensions of the open-topped box with volume 4000 cm^3 whose bottom and four sides have minimal total surface area.

In Problems 39 through 42, you are to find the dimensions that minimize the total cost of the material needed to construct the rectangular box that is described. It is either closed (top, bottom, and four sides) or open-topped (four sides and a bottom).

39. The box is to be open-topped with a volume of 600 in.^{3} The material for its bottom costs 6¢/in.^{2} and the material for its four sides costs 5¢/in.^{2}

40. The box is to be closed with a volume of 48 ft^3. The material for its top and bottom costs $\$3/\text{ft}^2$ and the material for its four sides costs $\$4/\text{ft}^2$.

41. The box is to be closed with a volume of 750 in.^3 The material for its top and bottom costs $3¢/\text{in.}^2$, the material for its front and back costs $6¢/\text{in.}^2$, and the material for its two ends costs $9¢/\text{in.}^2$

42. The box is to be a closed shipping crate with a volume of 12 m^3. The material for its bottom costs *twice* as much (per square meter) as the material for its top and four sides.

43. A rectangular building is to have a volume of 8000 ft^3. Annual heating and cooling costs will amount to $\$2/\text{ft}^2$ for its top, front, and back, and $\$4/\text{ft}^2$ for the two end walls. What dimensions of the building would minimize these annual costs?

44. You want to build a rectangular aquarium with a bottom made of slate costing $28¢/\text{in.}^2$ Its sides will be glass, which costs $5¢/\text{in.}^2$, and its top will be stainless steel, which costs $2¢/\text{in.}^2$ The volume of this aquarium is to be 24,000 in.^3 What are the dimensions of the least expensive such aquarium?

45. A rectangular box is inscribed in the first octant with three of its sides in the coordinate planes, their common vertex at the origin, and the opposite vertex on the plane with equation $x + 3y + 7z = 11$. What is the maximum possible volume of such a box?

46. Three sides of a rectangular box lie in the coordinate planes, their common vertex at the origin; the opposite vertex is on the plane with equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(a, b, and c are positive constants). In terms of a, b, and c, what is the maximum possible volume of such a box?

47. Find the maximum volume of a rectangular box that a post office will accept for delivery if the sum of its *length* and *girth* cannot exceed 108 in.

48. Repeat Problem 47 for the case of a cylindrical box—one shaped like a hatbox or a fat mailing tube.

49. A rectangular box with its base in the xy-plane is inscribed under the graph of the paraboloid $z = 1 - x^2 - y^2$, $z \geqq 0$. Find the maximum possible volume of the box. [*Suggestion:* You may assume that the sides of the box are parallel to the vertical coordinate planes, and it follows that the box is symmetrically placed around these planes.]

50. What is the maximum possible volume of a rectangular box inscribed in a hemisphere of radius R? Assume that one face of the box lies in the planar base of the hemisphere.

51. A buoy is to have the shape of a right circular cylinder capped at each end by identical right circular cones with the same radius as the cylinder. Find the minimum possible surface area of the buoy, given that it has fixed volume V.

52. A pentagonal window is to have the shape of a rectangle surmounted by an isosceles triangle (with horizontal base, so the window is symmetric around its vertical axis), and the perimeter of the window is to be 24 ft. What are the dimensions of such a window that will admit the most light (because its area is the greatest)?

53. Find the point (x, y) in the plane for which the sum of the squares of its distances from $(0, 1)$, $(0, 0)$, and $(2, 0)$ is a minimum.

54. Find the point (x, y) in the plane for which the sum of the squares of its distances from (a_1, b_1), (a_2, b_2), and (a_3, b_3) is a minimum.

55. An A-frame house is to have fixed volume V. Its front and rear walls are in the shape of equal, parallel isosceles triangles with horizontal bases. The roof consists of two rectangles that connect pairs of upper sides of the triangles. To minimize heating and cooling costs, the total area of the A-frame (excluding the floor) is to be minimized. Describe the shape of the A-frame of minimal area.

56. What is the maximum possible volume of a rectangular box whose longest diagonal has fixed length L?

57. A wire 120 cm long is cut into three *or fewer* pieces, and each piece is bent into the shape of a square. How should this be done to minimize the total area of these squares? To maximize it?

58. You must divide a lump of putty of fixed volume V into three or fewer pieces and form the pieces into cubes. How should you do this to maximize the total surface area of the cubes? To minimize it?

59. A very long rectangle of sheet metal has width L and is to be folded to make a rain gutter (Fig. 12.5.16). Maximize its volume by maximizing the cross-sectional area shown in the figure.

FIGURE 12.5.16 Cross section of the rain gutter of Problem 59.

60. Consider the function $f(x, y) = (y - x^2)(y - 3x^2)$. (a) Show that $f_x(0, 0) = 0 = f_y(0, 0)$. (b) Show that for every straight line $y = mx$ through $(0, 0)$, the function $f(x, mx)$ has a local minimum at $x = 0$. (c) Examine the values of f at points of the parabola $y = 2x^2$ to show that f does *not* have a local minimum at $(0, 0)$. This tells us that we cannot use the line-through-the-point method of Example 8 to show that a point *is* a local extremum.

61. Suppose that Alpha, Inc. and Beta, Ltd. manufacture competitive (but not identical) products, with the weekly sales of each product determined by the selling price of that product *and* the price of its competition. Suppose that Alpha sets a sales price of x dollars per unit for its product, while Beta sets a sales price of y dollars per unit for its product. Market research shows that the weekly profit made by Alpha is then

$$P(x) = -2x^2 + 12x + xy - y - 10$$

and that the weekly profit made by Beta is

$$Q(y) = -3y^2 + 18y + 2xy - 2x - 15$$

(both in thousands of dollars). The peculiar notation arises from the fact that x is the only variable under the control of Alpha and y is the only variable under the control of Beta. (If this disturbs you, feel free to write $P(x, y)$ in place of $P(x)$ and $Q(x, y)$ in place of $Q(y)$.) (a) Assume that both company managers know calculus and that each knows that the *other* knows calculus and has some common sense.

What price will each manager set to maximize his company's weekly profit? (b) Now suppose that the two managers enter into an agreement (legal or otherwise) by which they plan to maximize their *total* weekly profit. Now what should be the selling price of each product? (We suppose that they will divide the resulting profit in an equitable way, but the details of this intriguing problem are not the issue.)

62. Three firms—Ajax Products (AP), Behemoth Quicksilver (BQ), and Conglomerate Resources (CR)—produce products in quantities A, B, and C, respectively. The weekly profits that accrue to each, in thousands of dollars, obey the following equations:

$$\begin{aligned} \text{AP:} \quad & P = 1000A - A^2 - 2AB, \\ \text{BQ:} \quad & Q = 2000B - 2B^2 - 4BC, \\ \text{CR:} \quad & R = 1500C - 3C^2 - 6AC. \end{aligned}$$

(a) If each firm acts independently to maximize its weekly profit, what will those profits be? (b) If firms AP and CR join to maximize their total profit while BQ continues to act alone, what effects will this have? Give a *complete* answer to this problem. Assume that the fact of the merger of AP and CR is known to the management of BQ.

63. A farmer can raise sheep, hogs, and cattle. She has space for 80 sheep or 120 hogs or 60 cattle or any combination using the same amount of space; that is, 8 sheep use as much space as 12 hogs or 6 cattle. The anticipated profits per animal are \$10 per sheep, \$8 per hog, and \$20 for each head of cattle. State law requires that a farmer raise as many hogs as sheep and cattle combined. How does the farmer maximize her profit?

Problems 64 and 65 deal with the quadratic form

$$f(x, y) = ax^2 + 2bxy + cy^2. \tag{9}$$

64. Show that the quadratic form f in (9) has only the single critical point $(0, 0)$ unless $ac - b^2 = 0$, in which case every point on a certain line through the origin is a critical point. Experiment with computer graphs to formulate a conjecture about the shape of the surface $z = f(x, y)$ in the exceptional case $ac - b^2 = 0$. Can you substantiate your conjecture?

65. Use a computer algebra system to graph the quadratic form in (9) for a variety of different values of the coefficients a, b, and c in order to corroborate the following two conclusions. (a) If $ac - b^2 > 0$, then the graph of $z = f(x, y)$ is an elliptic paraboloid and f therefore has either a maximum or a minimum value at $(0, 0)$. (b) If $ac - b^2 < 0$, then the graph of $z = f(x, y)$ is a hyperbolic paraboloid and f therefore has a saddle point at $(0, 0)$.

Figures 12.5.7 and 12.5.8 illustrate (and Problems 66 and 67 deal with) the cases $b = -\frac{1}{2}$ and $b = -\frac{3}{2}$ (respectively) of the special quartic form

$$f(x, y) = x^4 + 2bx^2y^2 + y^4. \tag{10}$$

66. Show that the quartic form f in (10) has only the single critical point $(0, 0)$ unless $b = -1$, in which case every point on a certain pair of lines through the origin is a critical point (Fig. 12.5.17). Experiment with computer graphs to formulate a conjecture about the shape of the graph of f in each of the two cases $b > -1$ and $b < -1$.

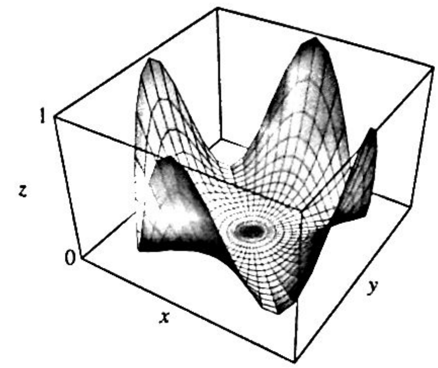

FIGURE 12.5.17 The graph of the function $f(x, y) = x^4 - x^2y^2 + y^4$ having critical points on the lines $y = \pm x$.

67. To show that the quartic form in (10) has a local minimum at the origin if $b > -1$ and a saddle point if $b < -1$, substitute $x = r\cos\theta$, $y = r\sin\theta$ and write $x^4 + 2bx^2y^2 + y^4 = r^4g(\theta)$. Then find the maximum and minimum values of $g(\theta)$ for $0 \leqq \theta \leqq 2\pi$.

68. Find the global maximum and minimum values of

$$f(x, y, z) = x^2 - 6xy + y^2 + 2yz + z^2 + 12.$$

What happens at the point or points at which all three partial derivatives of f are simultaneously zero?

69. Find the global maximum and minimum values of

$$g(x, y, z) = x^4 - 8x^2y^2 + y^4 + z^4 + 12.$$

What happens at the point or points at which all three partial derivatives of g are simultaneously zero?

70. The plane $\mathcal{P}$ with equation $x+y+z = 1$ meets the first octant in the triangle T for which x, y, and z are all non-negative. Find the maximum value of the expression $E = x - y + z$ on T. You will probably proceed by solving the equation of the plane $\mathcal{P}$ for $z = 1 - x - y$ and substituting for z in the expression E to obtain the quantity $h(x, y) = x - y + (1 - x - y)$ to be maximized. What happens at the point or points at which both partial derivatives of h are simultaneously zero?