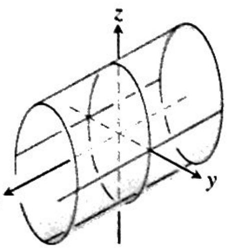
Exercises 12.6

Matching Equations with Surfaces

In Exercises 1–12, match the equation with the surface it defines. Also, identify each surface by type (paraboloid, ellipsoid, etc.) The surfaces are labeled (a)–(l).

1. $x^2 + y^2 + 4z^2 = 10$
2. $z^2 + 4y^2 - 4x^2 = 4$
3. $9y^2 + z^2 = 16$
4. $y^2 + z^2 = x^2$
5. $x = y^2 - z^2$
6. $x = -y^2 - z^2$
7. $x^2 + 2z^2 = 8$
8. $z^2 + x^2 - y^2 = 1$
9. $x = z^2 - y^2$
10. $z = -4x^2 - y^2$
11. $x^2 + 4z^2 = y^2$
12. $9x^2 + 4y^2 + 2z^2 = 36$

a.



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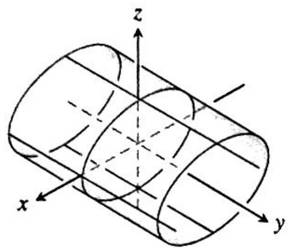

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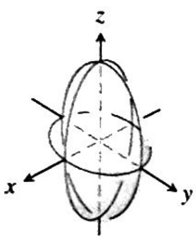

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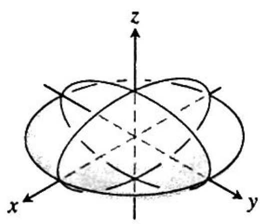
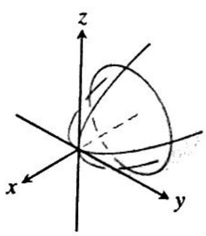
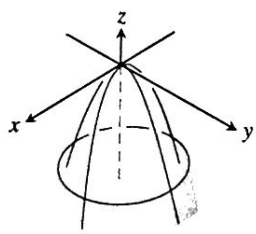


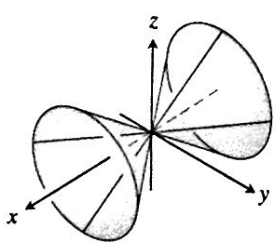
e.



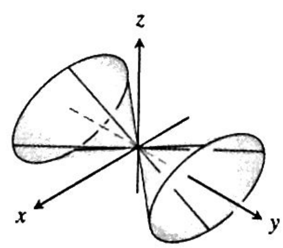
f.



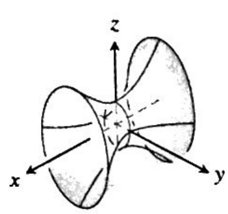
g.



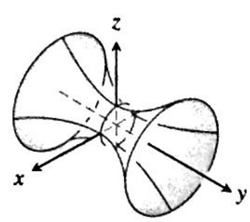
h.



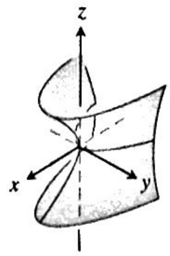
i.



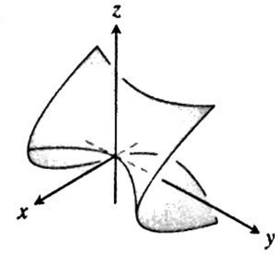
j.



k.



l.



Drawing

Sketch the surfaces in Exercises 13–44.

CYLINDERS

13. $x^2 + y^2 = 4$
14. $z = y^2 - 1$
15. $x^2 + 4z^2 = 16$
16. $4x^2 + y^2 = 36$

ELLIPSOIDS

17. $9x^2 + y^2 + z^2 = 9$
18. $4x^2 + 4y^2 + z^2 = 16$
19. $4x^2 + 9y^2 + 4z^2 = 36$
20. $9x^2 + 4y^2 + 36z^2 = 36$

PARABOLOIDS AND CONES

21. $z = x^2 + 4y^2$
22. $z = 8 - x^2 - y^2$
23. $x = 4 - 4y^2 - z^2$
24. $y = 1 - x^2 - z^2$
25. $x^2 + y^2 = z^2$
26. $4x^2 + 9z^2 = 9y^2$

HYPERBOLOIDS

27. $x^2 + y^2 - z^2 = 1$
28. $y^2 + z^2 - x^2 = 1$
29. $z^2 - x^2 - y^2 = 1$
30. $(y^2/4) - (x^2/4) - z^2 = 1$

HYPERBOLIC PARABOLOIDS

31. $y^2 - x^2 = z$
32. $x^2 - y^2 = z$

ASSORTED

33. $z = 1 + y^2 - x^2$
34. $4x^2 + 4y^2 = z^2$
35. $y = -(x^2 + z^2)$
36. $16x^2 + 4y^2 = 1$
37. $x^2 + y^2 - z^2 = 4$
38. $x^2 + z^2 = y$
39. $x^2 + z^2 = 1$
40. $16y^2 + 9z^2 = 4x^2$
41. $z = -(x^2 + y^2)$
42. $y^2 - x^2 - z^2 = 1$
43. $4y^2 + z^2 - 4x^2 = 4$
44. $x^2 + y^2 = z$

Theory and Examples

45. **a.** Express the area A of the cross-section cut from the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

by the plane $z = c$ as a function of c. (The area of an ellipse with semiaxes a and b is πab.)

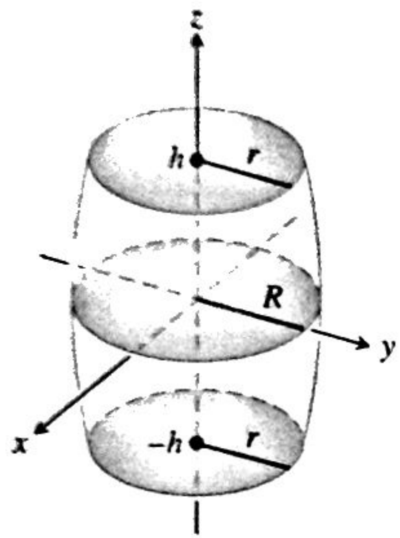
b. Use slices perpendicular to the z-axis to find the volume of the ellipsoid in part (a).

c. Now find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Does your formula give the volume of a sphere of radius a if $a = b = c$?

46. The barrel shown here is shaped like an ellipsoid with equal pieces cut from the ends by planes perpendicular to the z-axis. The cross-sections perpendicular to the z-axis are circular. The barrel is $2h$ units high, its midsection radius is R, and its end radii are both r. Find a formula for the barrel's volume. Then check two things. First, suppose the sides of the barrel are straightened to turn the barrel into a cylinder of radius R and height $2h$. Does your formula give the cylinder's volume? Second, suppose $r = 0$ and $h = R$ so the barrel is a sphere. Does your formula give the sphere's volume?



47. Show that the volume of the segment cut from the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

by the plane $z = h$ equals half the segment's base times its altitude.

48. a. Find the volume of the solid bounded by the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and the planes $z = 0$ and $z = h$, $h > 0$.

b. Express your answer in part (a) in terms of h and the areas A_0 and A_h of the regions cut by the hyperboloid from the planes $z = 0$ and $z = h$.

c. Show that the volume in part (a) is also given by the formula

$$V = \frac{h}{6}(A_0 + 4A_m + A_h),$$

where A_m is the area of the region cut by the hyperboloid from the plane $z = h/2$.

Viewing Surfaces

T Plot the surfaces in Exercises 49–52 over the indicated domains. If you can, rotate the surface into different viewing positions.

49. $z = y^2, \quad -2 \le x \le 2, \quad -0.5 \le y \le 2$

50. $z = 1 - y^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

51. ✓ $z = x^2 + y^2, \quad -3 \le x \le 3, \quad -3 \le y \le 3$

52. $z = x^2 + 2y^2$ over

a. $-3 \le x \le 3, \quad -3 \le y \le 3$

b. $-1 \le x \le 1, \quad -2 \le y \le 3$

c. $-2 \le x \le 2, \quad -2 \le y \le 2$

d. $-2 \le x \le 2, \quad -1 \le y \le 1$

COMPUTER EXPLORATIONS

Use a CAS to plot the surfaces in Exercises 53–58. Identify the type of quadric surface from your graph.

53. $\frac{x^2}{9} + \frac{y^2}{36} = 1 - \frac{z^2}{25}$

54. $\frac{x^2}{9} - \frac{z^2}{9} = 1 - \frac{y^2}{16}$

55. $5x^2 = z^2 - 3y^2$

56. $\frac{y^2}{16} = 1 - \frac{x^2}{9} + z$

57. $\frac{x^2}{9} - 1 = \frac{y^2}{16} + \frac{z^2}{2}$

58. $y - \sqrt{4 - z^2} = 0$

Please identify the graph of the quadratic equation on $\mathbb{R}^2$

$$144x^2 + 120xy + 25y^2 - 247x - 286y = 0$$

in the following way.

Step 1. Verify that the equation can be rewritten as $q(x, y) = 0$ with

$$q(x, y) = (x, y)A \begin{pmatrix} x \\ y \end{pmatrix} + (-247, -286) \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$A = \begin{pmatrix} 144 & 60 \\ 60 & 25 \end{pmatrix}.$$

Step 2. Solve the eigenvalue problem on the matrix A, that is to find all the eigenvalues and eigenvectors. Precisely, to find eigenvalues of A, we need to solve the characteristic equation of A:

$$\det(A - \lambda I) = 0,$$

where I stands for the 2×2 identity matrix. Suppose that the solution is denoted by $\lambda = \lambda_1$ or λ_2. For λ_1, to find the associated column eigenvector $\mathbf{v}_1 \neq 0$, solve the linear system

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad i.e.\ (A - \lambda_1 I)\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and for λ_2, to find the associated column eigenvector $\mathbf{v}_2 \neq 0$, solve the linear system

$$A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2, \quad i.e.\ (A - \lambda_2 I)\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Step 3. Define the rotation matrix R in the following way that if $\det(\mathbf{v}_1/|\mathbf{v}_1|, \mathbf{v}_2/|\mathbf{v}_2|) = 1$ then $R = (\mathbf{v}_1/|\mathbf{v}_1|, \mathbf{v}_2/|\mathbf{v}_2|)$, and if $\det(\mathbf{v}_1/|\mathbf{v}_1|, \mathbf{v}_2/|\mathbf{v}_2|) = -1$ then $R = (\mathbf{v}_1/|\mathbf{v}_1|, -\mathbf{v}_2/|\mathbf{v}_2|)$. Verify that R can be chosen as

$$R = \frac{1}{13} \begin{pmatrix} 12 & -5 \\ -5 & 12 \end{pmatrix}.$$

Step 4. Verify that by substituting

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{12}{13} & -\frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

into $q(x, y)$, one has the new form $\tilde{q}(u, v)$ without uv term, where

$$\tilde{q}(u, v) = 169u^2 - 338u - 169v.$$

Moreover, verify that in terms of uv coordinates, $q(x, y) = 0$ can be written as

$$u^2 - 2u - v = 0.$$

Step 5. Identify the graph of $u^2 - 2u - v = 0$ by drawing it on plane in terms of both the uv coordinate and xy coordinate.

Exercises 13.1

Motion in the Plane

In Exercises 1–4, $\mathbf{r}(t)$ is the position of a particle in the xy-plane at time t. Find an equation in x and y whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

1. $\mathbf{r}(t) = (t + 1)\mathbf{i} + (t^2 - 1)\mathbf{j}, \quad t = 1$

2. $\mathbf{r}(t) = \dfrac{t}{t + 1}\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad t = -1/2$

3. $\mathbf{r}(t) = e^t\mathbf{i} + \dfrac{2}{9}e^{2t}\mathbf{j}, \quad t = \ln 3$

4. $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (3 \sin 2t)\mathbf{j}, \quad t = 0$

Exercises 5–8 give the position vectors of particles moving along various curves in the xy-plane. In each case, find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve.

5. Motion on the circle $x^2 + y^2 = 1$

$$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \quad t = \pi/4 \text{ and } \pi/2$$

6. Motion on the circle $x^2 + y^2 = 16$

$$\mathbf{r}(t) = \left(4\cos\frac{t}{2}\right)\mathbf{i} + \left(4\sin\frac{t}{2}\right)\mathbf{j}; \quad t = \pi \text{ and } 3\pi/2$$

7. Motion on the cycloid $x = t - \sin t, \quad y = 1 - \cos t$

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}; \quad t = \pi \text{ and } 3\pi/2$$

8. Motion on the parabola $y = x^2 + 1$

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}; \quad t = -1, 0, \text{ and } 1$$

Motion in Space

In Exercises 9–14, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

9. $\mathbf{r}(t) = (t + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$

10. $\mathbf{r}(t) = (1 + t)\mathbf{i} + \dfrac{t^2}{\sqrt{2}}\mathbf{j} + \dfrac{t^3}{3}\mathbf{k}, \quad t = 1$

11. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k}, \quad t = \pi/2$

12. $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \dfrac{4}{3}t\mathbf{k}, \quad t = \pi/6$

13. $\mathbf{r}(t) = (2\ln(t + 1))\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}, \quad t = 1$

14. $\mathbf{r}(t) = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k}, \quad t = 0$

In Exercises 15–18, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$.

15. $\mathbf{r}(t) = (3t + 1)\mathbf{i} + \sqrt{3}t\mathbf{j} + t^2\mathbf{k}$

16. $\mathbf{r}(t) = \left(\dfrac{\sqrt{2}}{2}t\right)\mathbf{i} + \left(\dfrac{\sqrt{2}}{2}t - 16t^2\right)\mathbf{j}$

17. $\mathbf{r}(t) = (\ln(t^2 + 1))\mathbf{i} + (\tan^{-1} t)\mathbf{j} + \sqrt{t^2 + 1}\,\mathbf{k}$

18. $\mathbf{r}(t) = \dfrac{4}{9}(1 + t)^{3/2}\mathbf{i} + \dfrac{4}{9}(1 - t)^{3/2}\mathbf{j} + \dfrac{1}{3}t\mathbf{k}$

Tangents to Curves

As mentioned in the text, the **tangent line** to a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\mathbf{v}(t_0)$, the curve's velocity vector at t_0. In Exercises 19–22, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$.

19. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}, \quad t_0 = 0$

20. $\mathbf{r}(t) = t^2\mathbf{i} + (2t - 1)\mathbf{j} + t^3\mathbf{k}, \quad t_0 = 2$

21. $\mathbf{r}(t) = \ln t\,\mathbf{i} + \dfrac{t - 1}{t + 2}\mathbf{j} + t\ln t\,\mathbf{k}, \quad t_0 = 1$

22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad t_0 = \dfrac{\pi}{2}$

Theory and Examples

23. **Motion along a circle** Each of the following equations in parts (a)–(e) describes the motion of a particle having the same path, namely the unit circle $x^2 + y^2 = 1$. Although the path of each particle in parts (a)–(e) is the same, the behavior, or "dynamics," of each particle is different. For each particle, answer the following questions.

 i) Does the particle have constant speed? If so, what is its constant speed?

 ii) Is the particle's acceleration vector always orthogonal to its velocity vector?

 iii) Does the particle move clockwise or counterclockwise around the circle?

 iv) Does the particle begin at the point $(1, 0)$?

 a. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad t \ge 0$

 b. $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}, \quad t \ge 0$

 c. $\mathbf{r}(t) = \cos(t - \pi/2)\mathbf{i} + \sin(t - \pi/2)\mathbf{j}, \quad t \ge 0$

 d. $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}, \quad t \ge 0$

 e. $\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j}, \quad t \ge 0$

24. **Motion along a circle** Show that the vector-valued function

$$\mathbf{r}(t) = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \cos t\left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) + \sin t\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$$

describes the motion of a particle moving in the circle of radius 1 centered at the point $(2, 2, 1)$ and lying in the plane $x + y - 2z = 2$.

25. **Motion along a parabola** A particle moves along the top of the parabola $y^2 = 2x$ from left to right at a constant speed of 5 units per second. Find the velocity of the particle as it moves through the point $(2, 2)$.

26. **Motion along a cycloid** A particle moves in the xy-plane in such a way that its position at time t is

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}.$$

 T a. Graph $\mathbf{r}(t)$. The resulting curve is a cycloid.

 b. Find the maximum and minimum values of $|\mathbf{v}|$ and $|\mathbf{a}|$. (*Hint:* Find the extreme values of $|\mathbf{v}|^2$ and $|\mathbf{a}|^2$ first and take square roots later.)

27. Let $\mathbf{r}$ be a differentiable vector function of t. Show that if $\mathbf{r}\cdot(d\mathbf{r}/dt) = 0$ for all t, then $|\mathbf{r}|$ is constant.

28. **Derivatives of triple scalar products**

 a. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable vector functions of t, then

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}.$$

 b. Show that

$$\frac{d}{dt}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right).$$

 (*Hint:* Differentiate on the left and look for vectors whose products are zero.)

29. Prove the two Scalar Multiple Rules for vector functions.

30. Prove the Sum and Difference Rules for vector functions.

31. **Component Test for Continuity at a Point** Show that the vector function $\mathbf{r}$ defined by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at $t = t_0$ if and only if f, g, and h are continuous at t_0.

32. **Limits of cross products of vector functions** Suppose that $\mathbf{r}_1(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, $\mathbf{r}_2(t) = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$, $\lim_{t\to t_0} \mathbf{r}_1(t) = \mathbf{A}$, and $\lim_{t\to t_0} \mathbf{r}_2(t) = \mathbf{B}$. Use the determinant formula for cross products and the Limit Product Rule for scalar functions to show that

$$\lim_{t\to t_0} (\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{A} \times \mathbf{B}.$$

33. **Differentiable vector functions are continuous** Show that if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is differentiable at $t = t_0$, then it is continuous at t_0 as well.

34. **Constant Function Rule** Prove that if $\mathbf{u}$ is the vector function with the constant value $\mathbf{C}$, then $d\mathbf{u}/dt = \mathbf{0}$.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 35–38.

a. Plot the space curve traced out by the position vector $\mathbf{r}$.

b. Find the components of the velocity vector $d\mathbf{r}/dt$.

c. Evaluate $d\mathbf{r}/dt$ at the given point t_0 and determine the equation of the tangent line to the curve at $\mathbf{r}(t_0)$.

d. Plot the tangent line together with the curve over the given interval.

35. $\mathbf{r}(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 6\pi$, $t_0 = 3\pi/2$

36. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $-2 \le t \le 3$, $t_0 = 1$

37. $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\ln(1 + t))\mathbf{j} + t\mathbf{k}$, $0 \le t \le 4\pi$, $t_0 = \pi/4$

38. $\mathbf{r}(t) = (\ln(t^2 + 2))\mathbf{i} + (\tan^{-1} 3t)\mathbf{j} + \sqrt{t^2 + 1}\,\mathbf{k}$, $-3 \le t \le 5$, $t_0 = 3$

In Exercises 39 and 40, you will explore graphically the behavior of the helix

$$\mathbf{r}(t) = (\cos at)\mathbf{i} + (\sin at)\mathbf{j} + bt\mathbf{k}$$

as you change the values of the constants a and b. Use a CAS to perform the steps in each exercise.

39. Set $b = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $a = 1$, 2, 4, and 6 over the interval $0 \le t \le 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as a increases through these positive values.

40. Set $a = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $b = 1/4$, $1/2$, 2, and 4 over the interval $0 \le t \le 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as b increases through these positive values.

Exercises 13.2

Integrating Vector-Valued Functions

Evaluate the integrals in Exercises 1–10.

1. $\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t + 1)\mathbf{k}]\, dt$

2. $\int_1^2 \left[(6 - 6t)\mathbf{i} + 3\sqrt{t}\,\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right] dt$

3. $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + (\sec^2 t)\mathbf{k}]\, dt$

4. $\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}]\, dt$

5. $\int_1^4 \left[\frac{1}{t}\mathbf{i} + \frac{1}{5 - t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\right] dt$

6. $\int_0^1 \left[\frac{2}{\sqrt{1-t^2}}\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\mathbf{k}\right] dt$

7. $\int_0^1 [te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}]\, dt$

8. $\int_1^{\ln 3} [te^t\mathbf{i} + e^t\mathbf{j} + \ln t\,\mathbf{k}]\, dt$

9. $\int_0^{\pi/2} [\cos t\,\mathbf{i} - \sin 2t\,\mathbf{j} + \sin^2 t\,\mathbf{k}]\, dt$

10. $\int_0^{\pi/4} [\sec t\,\mathbf{i} + \tan^2 t\,\mathbf{j} - t\sin t\,\mathbf{k}]\, dt$

Initial Value Problems

Solve the initial value problems in Exercises 11–16 for $\mathbf{r}$ as a vector function of t.

11. Differential equation: $\frac{d\mathbf{r}}{dt} = -t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$

    Initial condition: $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

12. Differential equation: $\frac{d\mathbf{r}}{dt} = (180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}$

    Initial condition: $\mathbf{r}(0) = 100\mathbf{j}$

13. Differential equation: $\frac{d\mathbf{r}}{dt} = \frac{3}{2}(t+1)^{1/2}\mathbf{i} + e^{-t}\mathbf{j} + \frac{1}{t+1}\mathbf{k}$

    Initial condition: $\mathbf{r}(0) = \mathbf{k}$

14. Differential equation: $\frac{d\mathbf{r}}{dt} = (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$

    Initial condition: $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$

15. Differential equation: $\frac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$

    Initial conditions: $\mathbf{r}(0) = 100\mathbf{k}$ and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = 8\mathbf{i} + 8\mathbf{j}$

16. Differential equation: $\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$

    Initial conditions: $\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$ and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{0}$

Motion Along a Straight Line

17. At time $t = 0$, a particle is located at the point $(1, 2, 3)$. It travels in a straight line to the point $(4, 1, 4)$, has speed 2 at $(1, 2, 3)$ and constant acceleration $3\mathbf{i} - \mathbf{j} + \mathbf{k}$. Find an equation for the position vector $\mathbf{r}(t)$ of the particle at time t.

18. A particle traveling in a straight line is located at the point $(1, -1, 2)$ and has speed 2 at time $t = 0$. The particle moves toward the point $(3, 0, 3)$ with constant acceleration $2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find its position vector $\mathbf{r}(t)$ at time t.

Projectile Motion

Projectile flights in the following exercises are to be treated as ideal unless stated otherwise. All launch angles are assumed to be measured from the horizontal. All projectiles are assumed to be launched from the origin over a horizontal surface unless stated otherwise.

19. **Travel time** A projectile is fired at a speed of 840 m/sec at an angle of 60°. How long will it take to get 21 km downrange?

20. **Finding muzzle speed** Find the muzzle speed of a gun whose maximum range is 24.5 km.

21. **Flight time and height** A projectile is fired with an initial speed of 500 m/sec at an angle of elevation of 45°.

    a. When and how far away will the projectile strike?

    b. How high overhead will the projectile be when it is 5 km downrange?

    c. What is the greatest height reached by the projectile?

22. **Throwing a baseball** A baseball is thrown from the stands 9.8 m above the field at an angle of 30° up from the horizontal. When and how far away will the ball strike the ground if its initial speed is 9.8 m/sec?

23. **Firing golf balls** A spring gun at ground level fires a golf ball at an angle of 45°. The ball lands 10 m away.

    a. What was the ball's initial speed?

    b. For the same initial speed, find the two firing angles that make the range 6 m.

24. **Beaming electrons** An electron in a TV tube is beamed horizontally at a speed of 5×10^6 m/sec toward the face of the tube 40 cm away. About how far will the electron drop before it hits?

25. **Equal-range firing angles** What two angles of elevation will enable a projectile to reach a target 16 km downrange on the same level as the gun if the projectile's initial speed is 400 m/sec?

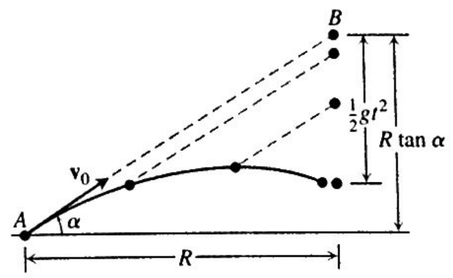
26. **Range and height versus speed**

    a. Show that doubling a projectile's initial speed at a given launch angle multiplies its range by 4.

    b. By about what percentage should you increase the initial speed to double the height and range?

27. Verify the results given in the text (following Example 4) for the maximum height, flight time, and range for ideal projectile motion.

28. **Colliding marbles** The accompanying figure shows an experiment with two marbles. Marble A was launched toward marble B with launch angle α and initial speed v_0. At the same instant, marble B was released to fall from rest at $R \tan \alpha$ units directly above a spot R units downrange from A. The marbles were found to collide regardless of the value of v_0. Was this mere coincidence, or must this happen? Give reasons for your answer.



29. **Firing from (x_0, y_0)** Derive the equations

$$x = x_0 + (v_0 \cos\alpha)t,$$

$$y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

(see Equation (7) in the text) by solving the following initial value problem for a vector $\mathbf{r}$ in the plane.

Differential equation: $\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}$

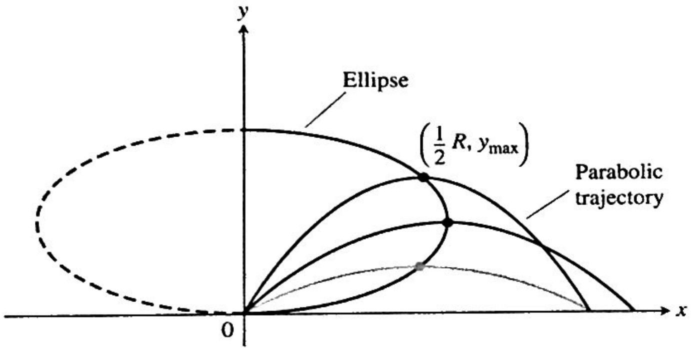
Initial conditions: $\mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j}$

$$\frac{d\mathbf{r}}{dt}(0) = (v_0 \cos\alpha)\mathbf{i} + (v_0 \sin\alpha)\mathbf{j}$$

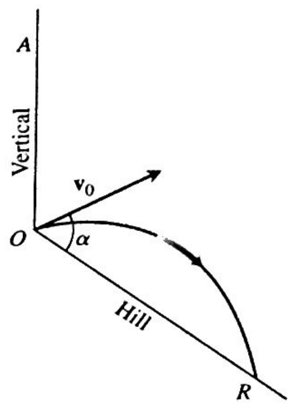
30. **Where trajectories crest** For a projectile fired from the ground at launch angle α with initial speed v_0, consider α as a variable and v_0 as a fixed constant. For each α, $0 < \alpha < \pi/2$, we obtain a parabolic trajectory as shown in the accompanying figure. Show that the points in the plane that give the maximum heights of these parabolic trajectories all lie on the ellipse

$$x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{4g^2},$$

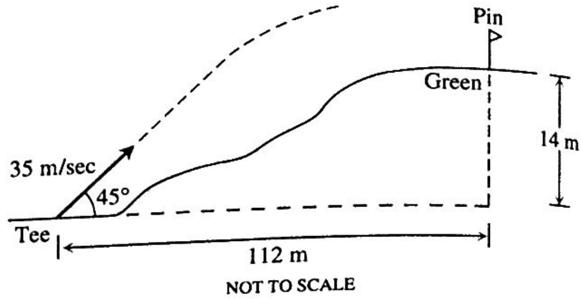
where $x \geq 0$.



31. **Launching downhill** An ideal projectile is launched straight down an inclined plane as shown in the accompanying figure.
   a. Show that the greatest downhill range is achieved when the initial velocity vector bisects angle AOR.
   b. If the projectile were fired uphill instead of down, what launch angle would maximize its range? Give reasons for your answer.



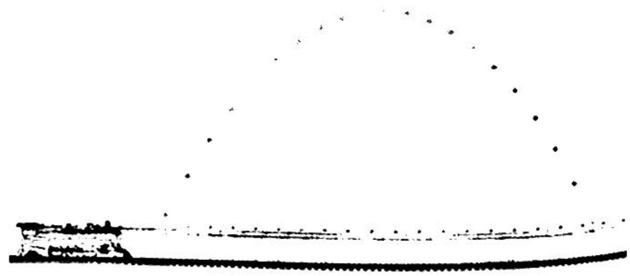
32. **Elevated green** A golf ball is hit with an initial speed of 35 m/sec at an angle of elevation of 45° from the tee to a green that is elevated 14 m above the tee as shown in the diagram. Assuming that the pin, 112 m downrange, does not get in the way, where will the ball land in relation to the pin?



33. **Volleyball** A volleyball is hit when it is 1.2 m above the ground and 3.7 m from a 2-m-high net. It leaves the point of impact with an initial velocity of 11 m/sec at an angle of 27° and slips by the opposing team untouched.
   a. Find a vector equation for the path of the volleyball.
   b. How high does the volleyball go, and when does it reach maximum height?
   c. Find its range and flight time.
   d. When is the volleyball 2.2 m above the ground? How far (ground distance) is the volleyball from where it will land?
   e. Suppose that the net is raised to 2.5 m. Does this change things? Explain.

34. **Shot put** In Moscow in 1987, Natalya Lisouskaya set a women's world record by putting a 39.2 N shot 22.5 m. Assuming that she launched the shot at a 40° angle to the horizontal from 2 m above the ground, what was the shot's initial speed?

35. **Model train** The accompanying multiflash photograph shows a model train engine moving at a constant speed on a straight horizontal track. As the engine moved along, a marble was fired into the air by a spring in the engine's smokestack. The marble, which continued to move with the same forward speed as the engine, rejoined the engine 1 sec after it was fired. Measure the angle the marble's path made with the horizontal and use the information to find how high the marble went and how fast the engine was moving.



36. **Hitting a baseball under a wind gust** A baseball is hit when it is 76 cm above the ground. It leaves the bat with an initial velocity of 44 m/sec at a launch angle of 23°. At the instant the ball is hit, an instantaneous gust of wind blows against the ball, adding a component of $-4.3\mathbf{i}$ (m/sec) to the ball's initial velocity. A 4.6-m-high fence lies 90 m from home plate in the direction of the flight.
   a. Find a vector equation for the path of the baseball.
   b. How high does the baseball go, and when does it reach maximum height?

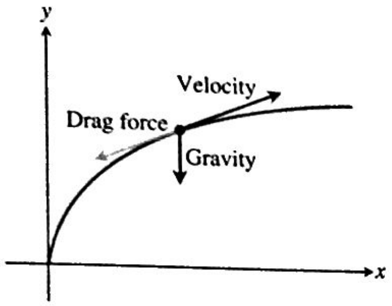
c. Find the range and flight time of the baseball, assuming that the ball is not caught.

d. When is the baseball 6 m high? How far (ground distance) is the baseball from home plate at that height?

e. Has the batter hit a home run? Explain.

Projectile Motion with Linear Drag

The main force affecting the motion of a projectile, other than gravity, is air resistance. This slowing down force is **drag force,** and it acts in a direction *opposite* to the velocity of the projectile (see accompanying figure). For projectiles moving through the air at relatively low speeds, however, the drag force is (very nearly) proportional to the speed (to the first power) and so is called **linear.**



37. Linear drag Derive the equations

$$x = \frac{v_0}{k}(1 - e^{-kt})\cos\alpha$$

$$y = \frac{v_0}{k}(1 - e^{-kt})(\sin\alpha) + \frac{g}{k^2}(1 - kt - e^{-kt})$$

by solving the following initial value problem for a vector **r** in the plane.

Differential equation: $\quad \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j} - k\mathbf{v} = -g\mathbf{j} - k\frac{d\mathbf{r}}{dt}$

Initial conditions: $\quad \mathbf{r}(0) = \mathbf{0}$

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{v}_0 = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j}$$

The **drag coefficient** k is a positive constant representing resistance due to air density, v_0 and α are the projectile's initial speed and launch angle, and g is the acceleration of gravity.

38. Hitting a baseball with linear drag Consider the baseball problem in Example 5 when there is linear drag (see Exercise 37). Assume a drag coefficient $k = 0.12$, but no gust of wind.

a. From Exercise 37, find a vector form for the path of the baseball.

b. How high does the baseball go, and when does it reach maximum height?

c. Find the range and flight time of the baseball.

d. When is the baseball 9 m high? How far (ground distance) is the baseball from home plate at that height?

e. A 3-m-high outfield fence is 104 m from home plate in the direction of the flight of the baseball. The outfielder can jump and catch any ball up to 3.35 m off the ground to stop it from going over the fence. Has the batter hit a home run?

Theory and Examples

39. Establish the following properties of integrable vector functions.

a. The *Constant Scalar Multiple Rule:*

$$\int_a^b k\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt \quad \text{(any scalar } k\text{)}$$

The *Rule for Negatives,*

$$\int_a^b (-\mathbf{r}(t))\,dt = -\int_a^b \mathbf{r}(t)\,dt,$$

is obtained by taking $k = -1$.

b. The *Sum and Difference Rules:*

$$\int_a^b (\mathbf{r}_1(t) \pm \mathbf{r}_2(t))\,dt = \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$$

c. The *Constant Vector Multiple Rules:*

$$\int_a^b \mathbf{C}\cdot\mathbf{r}(t)\,dt = \mathbf{C}\cdot\int_a^b \mathbf{r}(t)\,dt \quad \text{(any constant vector } \mathbf{C}\text{)}$$

and

$$\int_a^b \mathbf{C}\times\mathbf{r}(t)\,dt = \mathbf{C}\times\int_a^b \mathbf{r}(t)\,dt \quad \text{(any constant vector } \mathbf{C}\text{)}$$

40. Products of scalar and vector functions Suppose that the scalar function $u(t)$ and the vector function $\mathbf{r}(t)$ are both defined for $a \le t \le b$.

a. Show that $u\mathbf{r}$ is continuous on $[a, b]$ if u and $\mathbf{r}$ are continuous on $[a, b]$.

b. If u and $\mathbf{r}$ are both differentiable on $[a, b]$, show that $u\mathbf{r}$ is differentiable on $[a, b]$ and that

$$\frac{d}{dt}(u\mathbf{r}) = u\frac{d\mathbf{r}}{dt} + \mathbf{r}\frac{du}{dt}.$$

41. Antiderivatives of vector functions

a. Use Corollary 2 of the Mean Value Theorem for scalar functions to show that if two vector functions $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on an interval I, then the functions differ by a constant vector value throughout I.

b. Use the result in part (a) to show that if $\mathbf{R}(t)$ is any antiderivative of $\mathbf{r}(t)$ on I, then any other antiderivative of $\mathbf{r}$ on I equals $\mathbf{R}(t) + \mathbf{C}$ for some constant vector $\mathbf{C}$.

42. The Fundamental Theorem of Calculus The Fundamental Theorem of Calculus for scalar functions of a real variable holds for vector functions of a real variable as well. Prove this by using the theorem for scalar functions to show first that if a vector function $\mathbf{r}(t)$ is continuous for $a \le t \le b$, then

$$\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{r}(t)$$

at every point t of (a, b). Then use the conclusion in part (b) of Exercise 41 to show that if $\mathbf{R}$ is any antiderivative of $\mathbf{r}$ on $[a, b]$ then

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a).$$

43. Hitting a baseball with linear drag under a wind gust Consider again the baseball problem in Example 5. This time assume a drag coefficient of 0.08 *and* an instantaneous gust of wind that adds a component of $-5.4\mathbf{i}$ (m/sec) to the initial velocity at the instant the baseball is hit.

a. Find a vector equation for the path of the baseball.

b. How high does the baseball go, and when does it reach maximum height?

c. Find the range and flight time of the baseball.

d. When is the baseball 9 m high? How far (ground distance) is the baseball from home plate at that height?

e. A 6-m-high outfield fence is 116 m from home plate in the direction of the flight of the baseball. Has the batter hit a home run? If "yes," what change in the horizontal component of the ball's initial velocity would have kept the ball in the park? If "no," what change would have allowed it to be a home run?

44. Height versus time Show that a projectile attains three-quarters of its maximum height in half the time it takes to reach the maximum height.

Exercises 13.3

Finding Tangent Vectors and Lengths

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

1. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \le t \le \pi$
2. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \le t \le \pi$
3. $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le 8$
4. $\mathbf{r}(t) = (2 + t)\mathbf{i} - (t + 1)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 3$
5. $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \le t \le \pi/2$
6. $\mathbf{r}(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \quad 1 \le t \le 2$
7. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \left(2\sqrt{2}/3\right)t^{3/2}\mathbf{k}, \quad 0 \le t \le \pi$
8. $\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}, \quad \sqrt{2} \le t \le 2$
9. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

10. Find the point on the curve

$$\mathbf{r}(t) = (12\sin t)\mathbf{i} - (12\cos t)\mathbf{j} + 5t\mathbf{k}$$

at a distance 13π units along the curve from the point $(0, -12, 0)$ in the direction opposite to the direction of increasing arc length.

Arc Length Parameter

In Exercises 11–14, find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)|\,d\tau$$

from Equation (3). Then find the length of the indicated portion of the curve.

11. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le \pi/2$
12. $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \quad \pi/2 \le t \le \pi$
13. $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + e^t\mathbf{k}, \quad -\ln 4 \le t \le 0$
14. $\mathbf{r}(t) = (1 + 2t)\mathbf{i} + (1 + 3t)\mathbf{j} + (6 - 6t)\mathbf{k}, \quad -1 \le t \le 0$

Theory and Examples

15. **Arc length** Find the length of the curve

$$\mathbf{r}(t) = \left(\sqrt{2}t\right)\mathbf{i} + \left(\sqrt{2}t\right)\mathbf{j} + (1 - t^2)\mathbf{k}$$

from $(0, 0, 1)$ to $\left(\sqrt{2}, \sqrt{2}, 0\right)$.

16. **Length of helix** The length $2\pi\sqrt{2}$ of the turn of the helix in Example 1 is also the length of the diagonal of a square 2π units on a side. Show how to obtain this square by cutting away and flattening a portion of the cylinder around which the helix winds.

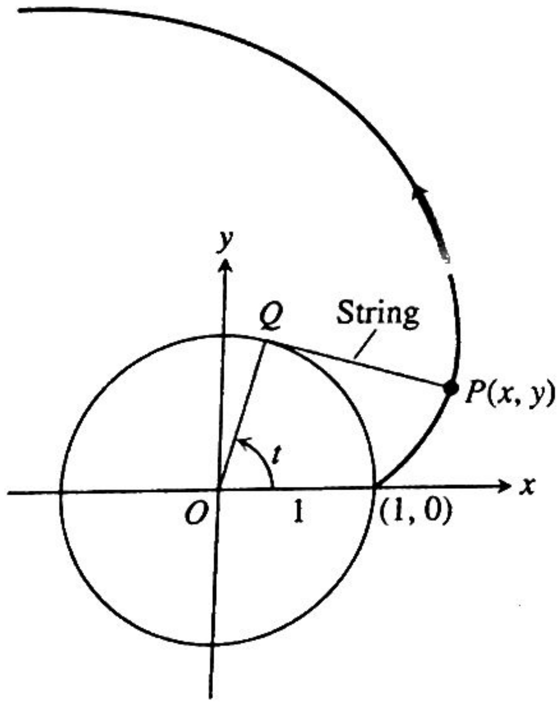
17. **Ellipse**
    a. Show that the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1 - \cos t)\mathbf{k}$, $0 \le t \le 2\pi$, is an ellipse by showing that it is the intersection of a right circular cylinder and a plane. Find equations for the cylinder and plane.
    b. Sketch the ellipse on the cylinder. Add to your sketch the unit tangent vectors at $t = 0, \pi/2, \pi$, and $3\pi/2$.
    c. Show that the acceleration vector always lies parallel to the plane (orthogonal to a vector normal to the plane). Thus, if you draw the acceleration as a vector attached to the ellipse, it will lie in the plane of the ellipse. Add the acceleration vectors for $t = 0, \pi/2, \pi$, and $3\pi/2$ to your sketch.
    d. Write an integral for the length of the ellipse. Do not try to evaluate the integral; it is nonelementary.
    e. **T** **Numerical integrator** Estimate the length of the ellipse to two decimal places.

18. **Length is independent of parametrization** To illustrate that the length of a smooth space curve does not depend on the parametrization you use to compute it, calculate the length of one turn of the helix in Example 1 with the following parametrizations.
    a. $\mathbf{r}(t) = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k}, \quad 0 \le t \le \pi/2$
    b. $\mathbf{r}(t) = [\cos(t/2)]\mathbf{i} + [\sin(t/2)]\mathbf{j} + (t/2)\mathbf{k}, \quad 0 \le t \le 4\pi$
    c. $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k}, \quad -2\pi \le t \le 0$

19. The involute of a circle If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an *involute* of the circle. In the accompanying figure, the circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at $(1, 0)$. The unwound portion of the string is tangent to the circle at Q, and t is the radian measure of the angle from the positive x-axis to segment OQ. Derive the parametric equations

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t > 0$$

of the point $P(x, y)$ for the involute.



20. (*Continuation of Exercise 19.*) Find the unit tangent vector to the involute of the circle at the point $P(x, y)$.

21. Distance along a line Show that if $\mathbf{u}$ is a unit vector, then the arc length parameter along the line $\mathbf{r}(t) = P_0 + t\mathbf{u}$ from the point $P_0(x_0, y_0, z_0)$ where $t = 0$, is t itself.

22. Use Simpson's Rule with $n = 10$ to approximate the length of arc of $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the origin to the point $(2, 4, 8)$.