THE CHINESE UNIVERSITY OF HONG KONG
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MATH2050C Mathematical Analysis I

Tutorial 9 (March 25)

Definition. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$.

- We say that f is **continuous at** c if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \delta$.
- Let $B \subseteq A$. We say that f is **continuous on** B if f is continuous at every point of B.

Remarks. (1) We do not assume that c is a cluster point of A.

Case 1: If $c \in A$ is a cluster point of A, then f is continuous at $c \iff \lim_{x\to c} f = f(c)$.

Case 2: If $c \in A$ is not a cluster point of A, then f is automatically continuous at c.

(2) "f is continuous on B" and "$f\big|_B$ is continuous" are different.

Example 1. (a) The function $g(x) \coloneqq \sin(1/x)$ for $x \neq 0$ does not have a limit at $x = 0$. Thus there is no value that we can assign at $x = 0$ to obtain a continuous extension of g at $x = 0$.

(b) Let $f(x) \coloneqq x \sin(1/x)$ for $x \neq 0$. If we define $F \colon \mathbb{R} \to \mathbb{R}$ by

$$F(x) \coloneqq \begin{cases} 0 & \text{for } x = 0, \\ x \sin(1/x) & \text{for } x \neq 0, \end{cases}$$

then F is continuous at $x = 0$.

Example 2. Show that the sine function is continuous on $\mathbb{R}$.

Suppose $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$.

Sequential Criterion for Continuity. *f is continuous at c if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

Discontinuity Criterion. *f is discontinuous at c if and only if there is a sequence (x_n) in A that converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.*

Example 3. Determine the points of continuity of the function $f(x) := [1/x]$, $x \neq 0$. Here $[\,\cdot\,]$ is the greatest integer function defined by

$$[x] := \sup\{n \in \mathbb{Z} : n \leq x\}.$$

Solution. First we show that f is discontinuous at each $1/m$, $m \in \mathbb{Z}\backslash\{0\}$. Let $x_n = \left(m - \frac{1}{2n}\right)^{-1}$ for $n \geq 1$. Then $\lim(x_n) = 1/m$. However, $f(x_n) = [m - \frac{1}{2n}] = m - 1$, so that

$$\lim f(x_n) = m - 1 \neq m = f(1/m).$$

By discontinuity criterion, f is discontinuous at $1/m$.

Next we show that f is continuous at each $c \in \mathbb{R}\backslash(\{0\} \cup \{1/m : m \in \mathbb{Z}\backslash\{0\}\})$. Observe that, $\delta := \min\{1/c - [1/c], [1/c] + 1 - 1/c\}/2$ satisfies

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \delta \implies \left[\frac{1}{x}\right] = \left[\frac{1}{c}\right].$$

Let $\varepsilon > 0$ be given. Take $\delta' := \min\{|c|/2, \delta|c|^2/2\}$. If $x \in \mathbb{R}\backslash\{0\}$ and $|x - c| < \delta'$, then $|x| > |c|/2$, and

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x||c|} < \frac{2}{|c|^2}|x - c| < \frac{2\delta'}{|c|^2} \leq \delta,$$

so that

$$|f(x) - f(c)| = \left|\left[\frac{1}{x}\right] - \left[\frac{1}{c}\right]\right| = 0 < \varepsilon.$$

Hence f is continuous at c. ◀

Classwork

1. Determine the points of continuity of the function $g(x) := x[x]$.

   Solution. First we show that g is discontinuous at each $m \in \mathbb{Z}\backslash\{0\}$. Let $x_n = m - \frac{1}{2n}$ for $n \geq 1$. Then $\lim(x_n) = m$. However, $g(x_n) = (m - \frac{1}{2n})(m-1)$, so that

   $$\lim g(x_n) = m(m-1) \neq m^2 = f(m).$$

   By discontinuity criterion, g is discontinuous at m.

   Next we show that g is continuous at 0. Note that

   $$g(x) = \begin{cases} -x & \text{if } -1 < x < 0 \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

   So $|g(x) - g(0)| \leq |x|$ for $x \in (-1, 1)$, and g is thus continuous at 0.

   Finally we show that g is continuous on $\bigcup_{m\in\mathbb{Z}}(m, m+1)$. Fix $c \in (m, m+1)$. Then

   $$|g(x) - g(c)| = |x \cdot m - c \cdot m| = |m||x - c| \quad \text{for } x \in (m, m+1).$$

   The continuity of g at c follows immediately. ◀

2. Give an example for each of the following:

   (a) $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere except at one point.

   (b) $f : \mathbb{R} \to \mathbb{R}$ is discontinuous everywhere.

   (c) $f : \mathbb{R} \to \mathbb{R}$ is continuous exactly at one point.

   (d) $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}\backslash\mathbb{Q}$ but distcontinuous on $\mathbb{Q}$.