THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 7 (March 11)

Definition (Contractive Sequences)**.** We say that a sequence (x_n) of real numbers is **contractive** if there exists a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#}$$

The number C is called the **constant** of the contractive sequence.

Remarks. Do not confuse (#) with the following condition:

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#\#}$$

For example, $(\sqrt{n})$ satisfies (##) but it is not contractive.

Theorem. *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Example 1. (Sequence of Fibonacci Fractions) Consider the sequence of Fibonacci fractions $x_n := f_n/f_{n+1}$, where (f_n) is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+2} := f_{n+1} + f_n$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $1/\varphi$, where $\varphi := (1+\sqrt{5})/2$ is the Golden Ratio.

Example 2. Let $Y = (y_n)$ be the sequence of real numbers given by

$$y_1 := \frac{1}{1!}, \quad y_2 := \frac{1}{1!} - \frac{1}{2!}, \quad \ldots \quad y_1 := \frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}, \quad \ldots$$

Show that $y := \lim(y_n)$ exists and $|y_n - y| \le \dfrac{1}{2^{n-1}}$ for all $n \in \mathbb{N}$.

Classwork

1. Let $x_n := \sqrt{n}$. Show that (x_n) satisfies $\lim |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence by definition.

   Solution. As (x_n) is clearly divergent, it cannot be contractive. However,

   $$|x_{n+2} - x_{n+1}| = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\sqrt{n+1} + \sqrt{n}} = |x_{n+1} - x_n|.$$

   ◀

2. Let (x_n) be a sequence of real numbers defined by

   $$\begin{cases} x_1 = 1, \quad x_2 = 2, \\ x_{n+2} := \dfrac{1}{3}(2x_{n+1} + x_n) \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

   Show that (x_n) is convergent and find its limit.

Solution. Note that

$$x_{n+2} - x_{n+1} = \frac{1}{3}(2x_{n+1} + x_n) - x_{n+1} = -\frac{1}{3}(x_{n+1} - x_n).$$

In particular,

$$|x_{n+2} - x_{n+1}| = \frac{1}{3}|x_{n+1} - x_n| \qquad \text{for } n \in \mathbb{N},$$

so (x_n) is contractive, hence convergent. As

$$x_{n+2} - x_{n+1} = -\frac{1}{3}(x_{n+1} - x_n) = \cdots = (-\frac{1}{3})^n(x_2 - x_1) = (-\frac{1}{3})^n.$$

we have

$$\sum_{k=0}^{n}(x_{k+2} - x_{k+1}) = \sum_{k=0}^{n}(-\frac{1}{3})^k$$
$$x_{n+2} - x_1 = \frac{1 - (-\frac{1}{3})^{n+1}}{1 - (-\frac{1}{3})}.$$

Hence $\lim(x_n) = \lim\left(1 + \frac{3}{4}(1 - (-\frac{1}{3})^{n+1})\right) = \frac{7}{4}$. ◀

3. If $x_1 > 0$ and $x_{n+1} \coloneqq (2 + x_n)^{-1}$ for $n \geq 1$, show that (x_n) is a convergent sequence. Find the limit.

Solution. By induction, it is easy to see that

$$0 \leq x_n \leq \frac{1}{2} \qquad \text{for } n \geq 2.$$

And so

$$\frac{2}{5} \leq \frac{1}{2 + x_n} \leq \frac{1}{2} \qquad \text{for } n \geq 2.$$

Now, for $n \geq 2$,

$$|x_{n+2} - x_{n+1}| = \frac{1}{(2 + x_n)(2 + x_{n+1})}|x_{n+1} - x_n| \leq \frac{1}{4}|x_{n+1} - x_n|.$$

So, the 1-tail of (x_n) is contractive, hence convergent. Thus (x_n) is also convergent. Suppose $x = \lim(x_n)$. Then we have $x = \dfrac{1}{2 + x}$, so that $x^2 + 2x - 1 = 0$. Solving the equation, we obtain $x = -1 + \sqrt{2}$ as the other root $-1 - \sqrt{2}$ is rejected. ◀