THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 6 (March 4)

Divergence Criteria. *If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.*

(i) X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.

(ii) X is not bounded.

Example 1. (a) Show that the sequence $X := ((-1)^n)$ is divergent.

(b) Show that the sequence $Y = (y_n) := \left(1, \frac{1}{2}, 3, \frac{1}{4}, \dots\right)$ is divergent.

(c) Show that the sequence $S := (\sin n)$ is divergent.

Example 2. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that (x_n) converges.

Solution. Recall that if (y_n) converges, then $(|y_n|)$ also converges. Since $x_n \geq 0$ for all n, we have $x_n = |x_n| = |(-1)^n x_n|$. Hence the convergence of (x_n) follows from the convergence of $((-1)^n x_n)$. ◀

Definition. Let $X = (x_n)$ be a bounded sequence of real numbers. Let

$$\mathcal{L} = \{\ell \in \mathbb{R} : \exists \text{ subseq } (x_{n_k}) \text{ of } (x_n) \text{ s.t. } (x_{n_k}) \to \ell\}.$$

The **limit superior** and **limit inferior** of (x_n) are defined, respectively, as

$$\limsup(x_n) = \overline{\lim}(x_n) := \sup \mathcal{L},$$
$$\liminf(x_n) = \underline{\lim}(x_n) := \inf \mathcal{L}.$$

Theorem. *(a) Let $u_m := \sup\{x_n : n \geq m\}$. Then (u_m) is decreasing and satisfies*

$$\limsup(x_n) = \lim(u_m) = \inf\{u_m : m \in \mathbb{N}\}.$$

(b) Let $v_m := \inf\{x_n : n \geq m\}$. Then (v_m) is increasing and satisfies

$$\liminf(x_n) = \lim(v_m) = \sup\{v_m : m \in \mathbb{N}\}.$$

Theorem. *For a real number x^*, $x^* = \limsup(x_n)$ if and only if given $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.*

Proof. ($\Longrightarrow$) Suppose $x^* = \limsup(x_n)$. Let $\varepsilon > 0$. Then $x^* + \varepsilon < x_n$ for at most a finite number of $n \in \mathbb{N}$. Otherwise, there exists $\ell \geq x^* + \varepsilon$ that is a subsequential limit of (x_n), contradicting $x^* = \sup \mathcal{L}$. On the other hand, there is a subsequence (x_{n_k}) of (x_n) that converges to some $\ell > x^* - \varepsilon$. In particular, $x^* - \varepsilon < x_n$ for an infinite number of $n \in \mathbb{N}$.

($\Longleftarrow$) Let $\varepsilon > 0$. Since $x^* + \varepsilon \geq x_n$ for all sufficiently large $n \in \mathbb{N}$, we have $x^* + \varepsilon \geq u_m$ for all sufficiently large $m \in \mathbb{N}$, and hence $x^* + \varepsilon \geq \limsup(x_n)$. On the other hand, $x^* - \varepsilon < x_n$ for an infinite number of $n \in \mathbb{N}$ implies that $x^* - \varepsilon < u_m$ for all $m \in \mathbb{N}$, so that $x^* - \varepsilon \leq \limsup(x_n)$. Since $\varepsilon > 0$ is arbitrary, we have $x^* - \limsup(x_n)$.

□

Example 3. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \dots).$$

Determine the values of $\limsup(x_n)$ and $\liminf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.

Solution. Observe that

$$v_m := \inf\{x_n : n \geq m\} = \begin{cases} x_{m+1} & m \text{ is odd} \\ x_m & m \text{ is even} \end{cases} = \begin{cases} -\frac{1}{(m+1)/2} & m \text{ is odd} \\ -\frac{1}{m/2} & m \text{ is even.} \end{cases}$$

Hence $\liminf(x_n) = \lim(v_m) = 0$.

Since $x_2 = -1$ is a lower bound of $\{x_n\}$, we have $\inf\{x_n\} = -1$.

Let $\varepsilon > 0$. Then $1 - \varepsilon < 1 < 1 + 1/n$ for all $n \in \mathbb{N}$, so that $1 - \varepsilon < x_n$ for an infinite number of $n \in \mathbb{N}$. On the other hand, note that $1 + \varepsilon > 1 > -1/n$ for $n \in \mathbb{N}$. And by choosing $N \in \mathbb{N}$ such that $1/N < \varepsilon$, we have $1 + \varepsilon > 1 + 1/N \geq 1 + /n$ for $n \geq N$. Thus $1 + \varepsilon \leq x_n$ for at most a finite number of $n \in \mathbb{N}$. Hence $\limsup(x_n) = 1$.

Since $x_1 = 2$ is an upper bound of $\{x_n\}$, we have $\sup\{x_n\} = 2$. ◀

Classwork

1. Prove that a bounded divergent sequence has two subsequences converging to different limits.

   Solution. Let (x_n) be a bounded divergent sequence. In particular, any subsequence of (x_n) is also bounded. By Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Suppose $\lim(x_{n_k}) = \ell$. Since (x_n) does not converge to ℓ, there are $\varepsilon_0 > 0$ and another subsequence (x_{m_k}) of (x_n) such that

   $$|x_{m_k} - \ell| \geq \varepsilon_0 \quad \text{for all } k \tag{\#}$$

   By Bolzano-Weierstrass Theorem again, (x_{m_k}) has a further subsequence $(x_{m_{k_j}})$ that converges to some real number ℓ'. By (#), $\ell \neq \ell'$. Now (x_{n_k}) and $(x_{m_{k_j}})$ are the desired subsequences of (x_n). ◀

2. Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\limsup(x_n) = \liminf(x_n)$.

   Solution. ($\implies$) Suppose $\lim(x_n) = \ell$. Then every subsequence of (x_n) converges to ℓ also. So $\mathcal{L} = \{\ell\}$. Hence $\limsup(x_n) = \sup \mathcal{L} = \ell$ and $\liminf(x_n) = \inf \mathcal{L} = \ell$.

   ($\impliedby$) Suppose $\limsup(x_n) = \ell = \liminf(x_n)$. Let $\varepsilon > 0$. Then $\lim(u_m) = \limsup(x_n) < \ell + \varepsilon$ implies that there is $N_1 \in \mathbb{N}$ such that $x_m \leq u_m < \ell + \varepsilon$ for $m \geq N_1$. Similarly, $\lim(v_m) = \liminf(x_n) > \ell - \varepsilon$ implies that there is $N_2 \in \mathbb{N}$ such that $x_m \geq v_m > \ell - \varepsilon$ for $m \geq N_2$. Now $\ell - \varepsilon < x_m < \ell + \varepsilon$ for $m \geq \max\{N_1, N_2\}$. Therefore $\lim(x_n) = \ell$. ◀