THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 4 (February 18)

Limit Theorems

Theorem 1. *Let $X = (x_n)$, $Y = (y_n)$ and $Z = (z_n)$ be sequences of real numbers that converge to x, y and z, respectively.*

(a) Let $c \in \mathbb{R}$. Then the sequences $X+Y, X-Y, X \cdot Y$, and cX converge to $x+y, x-y, xy$, and cx, respectively.

(b) Suppose further that $z_n \neq 0$ for all $n \in \mathbb{N}$, and $z \neq 0$. Then the sequence X/Z converges to x/z.

Example 1. Apply the above theorem to show the following limits.

(a) $\lim \left(\dfrac{2n+1}{n} \right) = 2.$

(b) $\lim \left(\dfrac{2n+1}{n+5} \right) = 2.$

(c) $\lim \left(\dfrac{2n}{n^2+1} \right) = 0.$

Theorem 2. *Let the sequence $X = (x_n)$ converge to x. Then the sequence $(|x_n|)$ of absolute values converges to $|x|$. That is, if $x = \lim(x_n)$, then $|x| = \lim(|x_n|)$.*

Theorem 3. *Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim(\sqrt{x_n}) = \sqrt{x}$.*

Classwork

1. If $a > 0$ and $b > 0$, show that $\lim\left(\sqrt{(n+a)(n+b)} - n\right) = (a+b)/2$.

2. Let (x_n) be a sequence of real numbers that converges to x. Show that, for any integer $m \geq 2$, $\lim(\sqrt[m]{|x_n|}) = \sqrt[m]{|x|}$.

   Solution. Let $m \geq 2$ be an integer. Note that, for $a, b \geq 0$, we have
   $$b^m - a^m = (b-a)(b^{m-1} + b^{m-2}a + b^{m-3}a^2 + \cdots + a^{m-1}),$$
   and hence
   $$|b^m - a^m| \geq a^{m-1}|b-a|.$$
   Thus,
   $$\left|\sqrt[m]{|x_n|} - \sqrt[m]{|x|}\right| \leq \begin{cases} \sqrt[m]{|x_n|} & \text{if } x = 0, \\ \dfrac{1}{(\sqrt[m]{|x|})^{m-1}}|x_n - x| & \text{if } x \neq 0. \end{cases}$$
   We can then argue as in the proof of Theorem 3 to show that $\lim(\sqrt[m]{|x_n|}) = \sqrt[m]{|x|}$.
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3. Let (x_n) be a sequence of real numbers. Define
   $$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \qquad \text{for all } n \in \mathbb{N}.$$
   If $\lim(x_n) = 0$, show that $\lim(s_n) = 0$.

   Solution. We separate s_n into two parts:
   $$s_n = \frac{x_1 + \cdots + x_m}{n} + \frac{x_{m+1} + \cdots + x_n}{n} \qquad \text{for } 1 \leq m < n.$$
   Since (x_n) is convergent, it is bounded, so we can find $M > 0$ such that
   $$|x_n| \leq M \qquad \text{for all } n \in \mathbb{N}.$$
   Let $\varepsilon > 0$ be given. Since $\lim(x_n) = 0$, there exists $m \in \mathbb{N}$ such that
   $$|x_n| < \varepsilon/2 \qquad \text{for all } n \geq m.$$
   By Archimedean Property, choose $N \in \mathbb{N}$ such that $N > \max\left\{\dfrac{mM}{\varepsilon/2}, m\right\}$.

   Now, for $n \geq N$, we have
   $$\begin{aligned} |s_n| &\leq \frac{|x_1| + \cdots + |x_m|}{n} + \frac{|x_{m+1}| + \cdots + |x_n|}{n} \\ &< \frac{mM}{n} + \frac{(n-m)\varepsilon/2}{n} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$
   Hence $\lim(s_n) = 0$. ◀