THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 11 (April 15)

Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u| \qquad \text{for all } x, u \in A, \tag{$*$}$$

then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

Remarks. When A is an interval I, the condition $(*)$ means that the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

Theorem. *If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Example 1. Show that

(a) $f(x) := x^2$ is a Lipschitz function on $[0, b]$, $b > 0$, but does not satisfy a Lipschitz condition on $[0, \infty)$.

(b) $g(x) := \sqrt{x}$ is uniformly continuous on $[0, 2]$ but not a Lipschitz function on $[0, 2]$.

(c) $g(x) := \sqrt{x}$ is a Lipschitz function on $[a, \infty)$, $a > 0$.

(d) $g(x) := \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Intermediate Value Theorem. *Let I be an interval and let $f \colon I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists $c \in I$ between a and b such that $f(c) = k$.*

Preservation of Interval Theorem. *Let I be an interval and let $f \colon I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.*

Classwork

1. Let f and g be Lipschitz functions on $[a, b]$. Show that the product fg is also a Lipschitz function on $[a, b]$.

   Solution. Since f and g are Lipschitz functions on $[a, b]$, there are $K_1, K_2 > 0$ such that

   $$\begin{cases} |f(x) - f(u)| \leq K_1|x - u| \\ |g(x) - g(u)| \leq K_2|x - u| \end{cases} \quad \text{for all } x, u \in [a, b].$$

   In particular, f and g are continuous on $[a, b]$. It follows from the Boundedness Theorem that there are $M_1, M_2 > 0$ such that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in [a, b]$. Now, for any $x, u \in [a, b]$, we have

   $$\begin{aligned} |f(x)g(x) - f(u)g(u)| &= |f(x)(g(x) - g(u)) + (f(x) - f(u))g(u)| \\ &\leq |f(x)||g(x) - g(u) + |f(x) - f(u)||g(u)| \\ &\leq M_1K_2|x - u| + M_2K_1|x - u| \\ &= (M_1K_2 + M_2K_1)|x - u|. \end{aligned}$$

   Hence fg is a Lipschitz function on $[a, b]$. ◀

2. Show that the polynomial $p(x) := x^4 + 7x^3 - 9$ has at least two real roots.

   Solution. Since p is continuous on $[0, 2]$ and $p(0) = -9 < 0 < 63 = p(2)$, it follows from the Intermediate Value Theorem that $p(c_1) = 0$ for some $c_1 \in (0, 2)$.

   Since p is continuous on $[-8, 0]$ and $p(-8) = 503 > 0 > -9 = p(0)$, it follows from the Intermediate Value Theorem that $p(c_2) = 0$ for some $c_2 \in (-8, 0)$.

   Since $(-8, 0) \cap (0, 2) = \emptyset$, $c_1 \neq c_2$. Hence p has at least two real roots. ◀