Week 9: Orthogonal Projections & Spectral Theorem (textbook § 6.6)
Unitary & Orthogonal Operators (textbook § 6.5)

Recall the Spectral Theorems:

Spectral Theorem: Let $T: V \to V$ be a linear operator on a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then,

there exists an orthonormal basis β for V s.t. $[T]_\beta$ is diagonal

$\Longleftrightarrow$ $\begin{cases} \text{when } \mathbb{F} = \mathbb{R},\ T \text{ is self adjoint, i.e. } \boxed{T^* = T}. \\ \text{when } \mathbb{F} = \mathbb{C},\ T \text{ is normal, i.e. } \boxed{T^*T = TT^*}. \end{cases}$

Question: What does it mean "geometrically" ?

Let us look at an example first with $\mathbb{F} = \mathbb{R}$.

Example: Consider the linear operator $T = L_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{i.e.} \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

Since $A^t = A \Rightarrow T$ is self-adjoint $\xRightarrow[\text{Thm.}]{\text{Spec.}}$ $\exists$ O.N.B. β which diagonalize T

How to find this β? Ans: Find eigenvalues / eigenvectors!

char. poly.: $\det(A - \lambda I) = \lambda^2 - 1 \Rightarrow$ Eigenvalues: $\boxed{\lambda_1 = 1,\ \lambda_2 = -1}$

Eigenspaces: $\begin{cases} E_{\lambda_1} = N(A - 1I) = N\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}. \\ E_{\lambda_2} = N(A - (-1)I) = N\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}. \end{cases}$

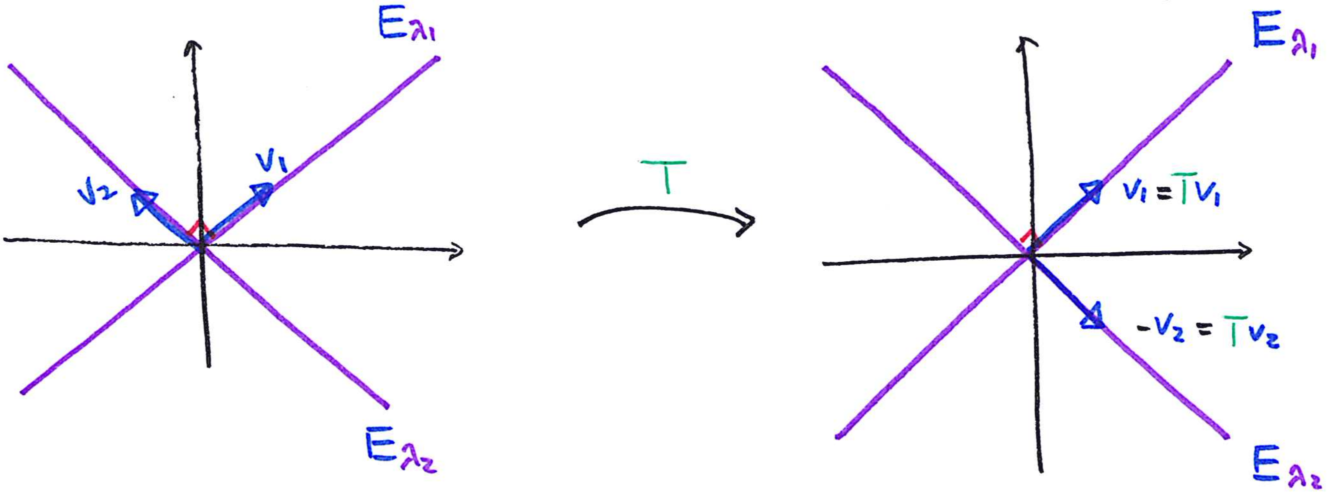
$\beta' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ is eigenbasis BUT NOT orthonormal!

Fortunately, β' is orthogonal (Q: why?) $\xRightarrow{\text{normalize}}$ $\beta = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ O.N.B.

which diagonalize T.

Note that $\boxed{E_{\lambda_1} \perp E_{\lambda_2}}$ and $\boxed{\mathbb{R}^2 = E_{\lambda_1} \oplus E_{\lambda_2}}$ "orthogonal decomposition"

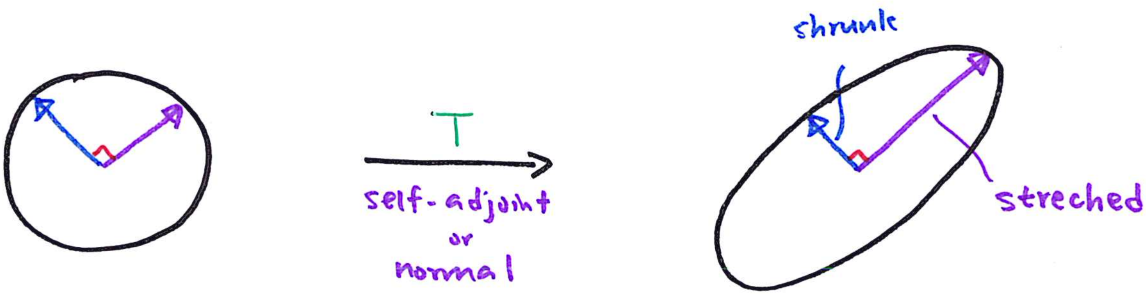
The "action" of T on each of these (T-invariant) subspaces are very simple: $\begin{cases} T(v_1) = v_1 & \text{for all } v_1 \in E_{\lambda_1} \\ T(v_2) = -v_2 & \text{for all } v_2 \in E_{\lambda_2} \end{cases}$



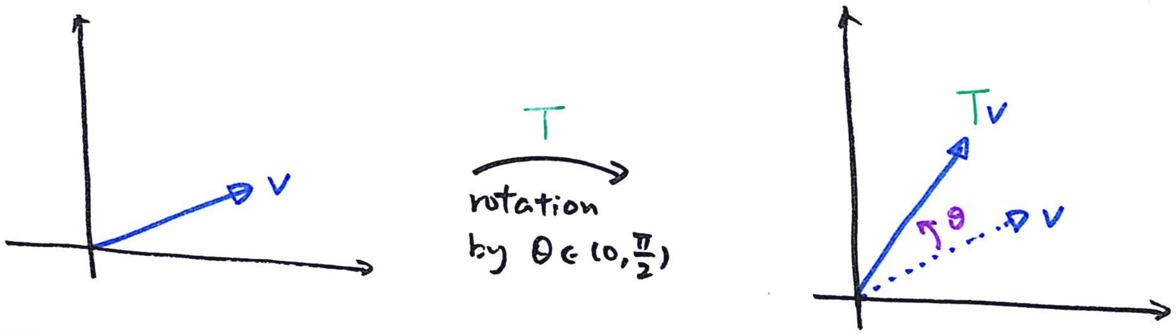
We understand the action of T by understanding its "sub-actions" on independent (orthogonal) directions. Thus, we can decompose T into its actions on the smaller orthogonal subspaces! (by rescaling)

Spectral Theorem $\Rightarrow$ we can carry out such decomposition for self adjoint / normal operators T

Geometrically, self-adjoint / normal operators T simply do some "stretching" and "shrinking" in different perpendicular directions!!



This is not true for nondiagonalizable operators, e.g:



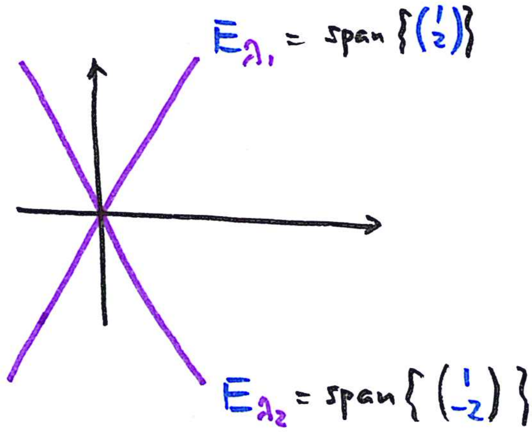
[There is no special "shrinking / stretching" directions!]

Even if T is diagonalizable, the "special shrinking / stretching" directions may <u>NOT</u> be $\perp$.

<u>Example</u>: $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$

whose eigenspaces are <u>not</u> orthogonal:



Only the normal / self adjoint operators have a "nice" <u>orthogonal</u> decomposition of V into its eigenspaces!

What about the "action" of T on vectors which do not lie on one of these eigen-directions?

<u>ANS</u>: Linearity!!

<u>Eg.</u> $\boxed{V = E_{\lambda_1} \oplus E_{\lambda_2}}$ orthogonal decomposition: $E_{\lambda_1} \perp E_{\lambda_2}$

ANY $v = v_1 + v_2 \Rightarrow Tv = Tv_1 + Tv_2$

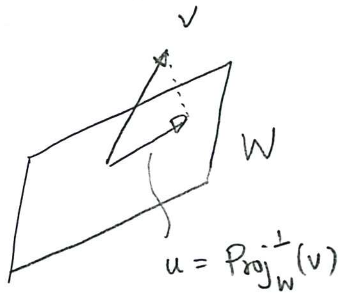
$$= \lambda_1 v_1 + \lambda_2 v_2$$

i.e.: the components of v are stretched/shrunk <u>separately</u> (decoupled!)

Orthogonal Projections

Recall that if $W \subset V$ is a finite dimensional subspace of an inner product space $(V, \langle , \rangle)$ – which could have $\dim V = +\infty$.

We have the orthogonal decomposition :



$$\boxed{V = W \oplus W^{\perp}}$$

Therefore, any $v \in V$ can be uniquely written as

$$\underset{W}{\underset{\cap}{v = u}} + \underset{W^{\perp}}{\underset{\cap}{z}}$$

, where $u = \mathrm{Proj}_W^{\perp}(v)$ is the orthogonal projection of v onto W.

In summary, for each $W \subset V$, we can define its orthogonal projection to be the map

$$\boxed{T = \mathrm{Proj}_W^{\perp} : V \to V}$$

Some observations :

(1) $\mathrm{Proj}_W^{\perp}$ is *linear*.

(2) $\boxed{R(T) = W}$ and $\boxed{N(T) = W^{\perp}}$

So, $R(T)^{\perp} = N(T)$ and $N(T)^{\perp} = R(T)$

(3) $\boxed{T^2 = T}$ since projecting the second time is redundant.

(4) $\boxed{T^* = T}$ If V is finite dimensional, then

$$[T]_\beta = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$$

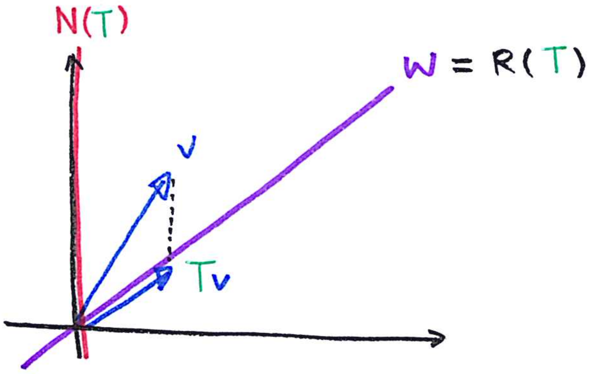
in some O.N.B. β which is clearly self adjoint!

↑

"standard projection matrix".

Question: Given a linear operator $T: V \to V$ on an inner product space V, when is T in fact an orthogonal projection onto some subspace $W \subset V$?

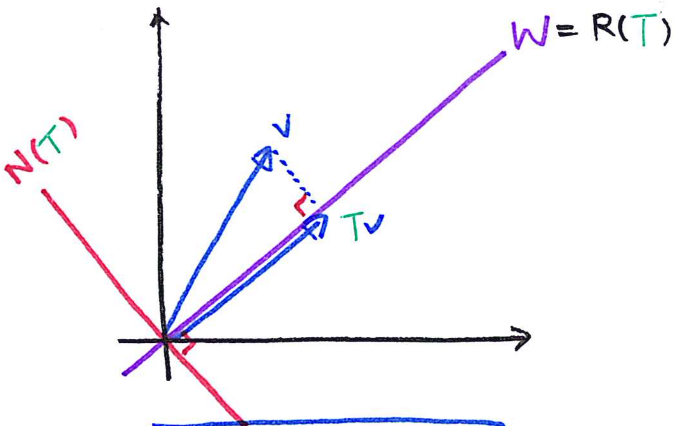
$\boxed{\text{Projections}}$ vs. $\boxed{\text{Orthogonal Projections}}$



$$\boxed{R(T)^{\perp} \neq N(T)}$$

but $V = R(T) \oplus N(T)$

$$T^2 = T$$



$$\boxed{R(T)^{\perp} = N(T)}$$

and $V = R(T) \oplus N(T)$

$$T^2 = T = T^* .$$

Def

Def$^{\underline{n}}$: Let $T: V \to V$ be a linear operator on an inner product space.

(i) T is a projection if $\boxed{T^2 = T}$

(ii) T is an orthogonal projection if $\boxed{T^2 = T}$ and

$$\boxed{R(T)^{\perp} = N(T), \ N(T)^{\perp} = R(T)}$$

Remark: An orthogonal projection T is most "efficient" that it satisfies a length decreasing property:

$$\boxed{\|Tv\| \leq \|v\| \quad \text{for } \underline{\text{all}} \ v \in V}$$

(Exercise: Give an example that this is NOT true for a general projection.)

Note: when $\dim V < +\infty$, $\boxed{R(T)^{\perp} = N(T)} \iff \boxed{N(T)^{\perp} = R(T)}$

since $(W^{\perp})^{\perp} = W$ for any finite dimensional subspace $W \subset V$.

The Proposition below justifies our definition of orthogonal projections.

Prop: Let $T : V \to V$ be a linear operator on an inner product space $(V, \langle , \rangle)$ with $\boxed{\dim V < +\infty}^{*}$. (*can be removed with slightly modified conclusions)

Then, the following are equivalent (TFAE):

(i) T is an orthogonal projection.

(ii) $\boxed{T^2 = T = T^*}$

(iii) There exists a subspace $W \subset V$ s.t. $\boxed{T = \mathrm{Proj}_W^{\perp}}$.

Proof: We will show that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii) : done before.

(ii) $\Rightarrow$ (i) : Need to check $R(T)^{\perp} = N(T)$.

In general, $\boxed{R(T)^{\perp} = N(T^*)}$ (Ex: Prove this!)

Therefore, we are done as $T = T^*$.

(i) $\Rightarrow$ (iii) : Assume $T^2 = T$ and $R(T)^{\perp} = N(T)$.

Define $W = R(T)$. We claim that $T = \mathrm{Proj}_W^{\perp}$.

$$R(T)^{\perp} = N(T) \Rightarrow V = \underbrace{\overset{W}{\overset{\parallel}{R(T)}} \oplus \overset{W^{\perp}}{\overset{\parallel}{N(T)}}}_{\text{orthogonal complements}}$$

It remains to show that

$$Ty = y \quad \text{for } \underline{\underline{\text{all}}} \ y \in R(T)$$

$$\iff \underbrace{T(Tx) = Tx}_{\text{true since } T^2 = T.} \quad \text{for } \underline{\underline{\text{all}}} \ x \in V$$

$\square$

Now, recall that the Spectral Theorems say that any normal / self adjoint operators $T : V \to V$ has an orthonormal eigenbasis β :

i.e.

$$[T]_\beta = \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}}^{E_{\lambda_1}} & & & \\ & \boxed{\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}}^{E_{\lambda_2}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix}\lambda_k & & \\ & \ddots & \\ & & \lambda_k\end{matrix}}^{E_{\lambda_k}} \end{pmatrix}$$

T acts independently by rescaling by λ_i on each eigenspace E_{λ_i}

Spectral Decomposition Theorem:

Let $(V, \langle , \rangle)$ be a finite dim. inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Suppose $T : V \to V$ is a normal ($\mathbb{F} = \mathbb{C}$) or self adjoint ($\mathbb{F} = \mathbb{R}$) operator. Denote the eigenvalues of T by

$$\lambda_1, \lambda_2, \dots, \lambda_k \qquad (\text{spectrum of } T)$$

Then, V has an orthogonal decomposition into its eigenspaces:

$$\boxed{V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}} \quad \text{where } E_{\lambda_i} \perp E_{\lambda_j} \ \ i \neq j$$

and T has a spectral decomposition into orthogonal projections:

$$\boxed{T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k} \quad \text{where } T_i = \mathrm{Proj}^{\perp}_{E_{\lambda_i}}$$

Proof: Just rephrasing the Spectral Theorems (see textbook Thm. 6.25). $\square$

! The Spectral Decomposition Theorem has surprisingly many interesting applications!.

Because it says we can decompose any normal / self adjoint operators into orthogonal projections - which is much simpler to understand.

Corollary 1: $\boxed{g(T) = g(\lambda_1)T_1 + g(\lambda_2)T_2 + \cdots + g(\lambda_k)T_k}$

for any polynomial g.

Example:

$$[T]_\beta = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} \Rightarrow [T^k]_\beta = \begin{pmatrix} \lambda_1^k I & 0 \\ 0 & \lambda_2^k I \end{pmatrix}$$

(Exercise: Can you prove the general case?)

Corollary 2: When $\mathbb{F} = \mathbb{C}$, T normal $(TT^* = T^*T)$ $\Leftrightarrow$ $T^* = g(T)$ for some polynomial g.

Proof: "$\Leftarrow$" trivial since T commutes with $g(T)$ for any polynomial g, e.g. $T(T^2+2T) = (T^2+2T)T$.

"$\Rightarrow$" Assume T is normal, then we have spectral decomposition

$$T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$$

$$\rightsquigarrow T^* = \bar{\lambda}_1 T_1 + \bar{\lambda}_2 T_2 + \cdots + \bar{\lambda}_k T_k \quad (\text{since } T_i^* = T_i)$$

Choose a polynomial g st $\boxed{g(\lambda_i) = \bar{\lambda}_i \text{ for all } i}$

- which can be done by Lagrange interpolation formula.

Then, we have by Corollary 1,

$$g(T) = g(\lambda_1)T_1 + \cdots + g(\lambda_k)T_k$$

$$= \bar{\lambda}_1 T_1 + \cdots + \bar{\lambda}_k T_k = T^*$$

$\square$

By a similar argument, one can show

Corollary 3: $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$

$\Rightarrow T_i = g_i(T)$ for some polynomial g_i

Corollary 4: Suppose $\mathbb{F} = \mathbb{C}$ and T is normal. Then

T is self adjoint $\iff$ all eigenvalues of T are real.

Proof: "$\Rightarrow$" proved before.

"$\Leftarrow$" Take $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$ where $\boxed{\lambda_i \in \mathbb{R}}$

$$\rightsquigarrow \quad T^* = \bar{\lambda}_1 T_1 + \cdots + \bar{\lambda}_k T_k$$
$$= \lambda_1 T_1 + \cdots + \lambda_k T_k = T$$

i.e. T is self adjoint! $\square$

> When our space has extra structure ... we have new concepts!

$\boxed{\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}}$	Vector Spaces	Inner Product Spaces
	$(V, +, \cdot)$	$(V, +, \cdot)$ & $\langle\, , \rangle$
model:	$\mathbb{R}^n$ or $\mathbb{C}^n$	$\mathbb{R}^n$ or $\mathbb{C}^n$ with $\langle\, , \rangle_{std}$
basis:	basis	orthonormal basis
"morphisms": or transformations	$T: V \to V$ linear (preserves $+$ & $\cdot$) $\boxed{T(ax+by) = aTx + bTy}$	$T: V \to V$ linear isometry (preserves $+, \cdot$ & $\langle\, , \rangle$) $\boxed{\langle Tx, Ty \rangle = \langle x, y \rangle}$ ($\mathbb{F}=\mathbb{C}$) / ($\mathbb{F}=\mathbb{R}$) unitary / orthogonal operators
		$\lVert \cdot \rVert$, $x \perp y$, T^*
change of basis:	invertible Q $[T]_\gamma = Q^{-1} [T]_\beta Q$	($\mathbb{F}=\mathbb{C}$) / ($\mathbb{F}=\mathbb{R}$) unitary / orthogonal Q $[T]_\gamma = Q^* [T]_\beta Q$
diagonalization:	eigenbasis	orthonormal eigenbasis

Orthogonal Operators on $\mathbb{R}^2$

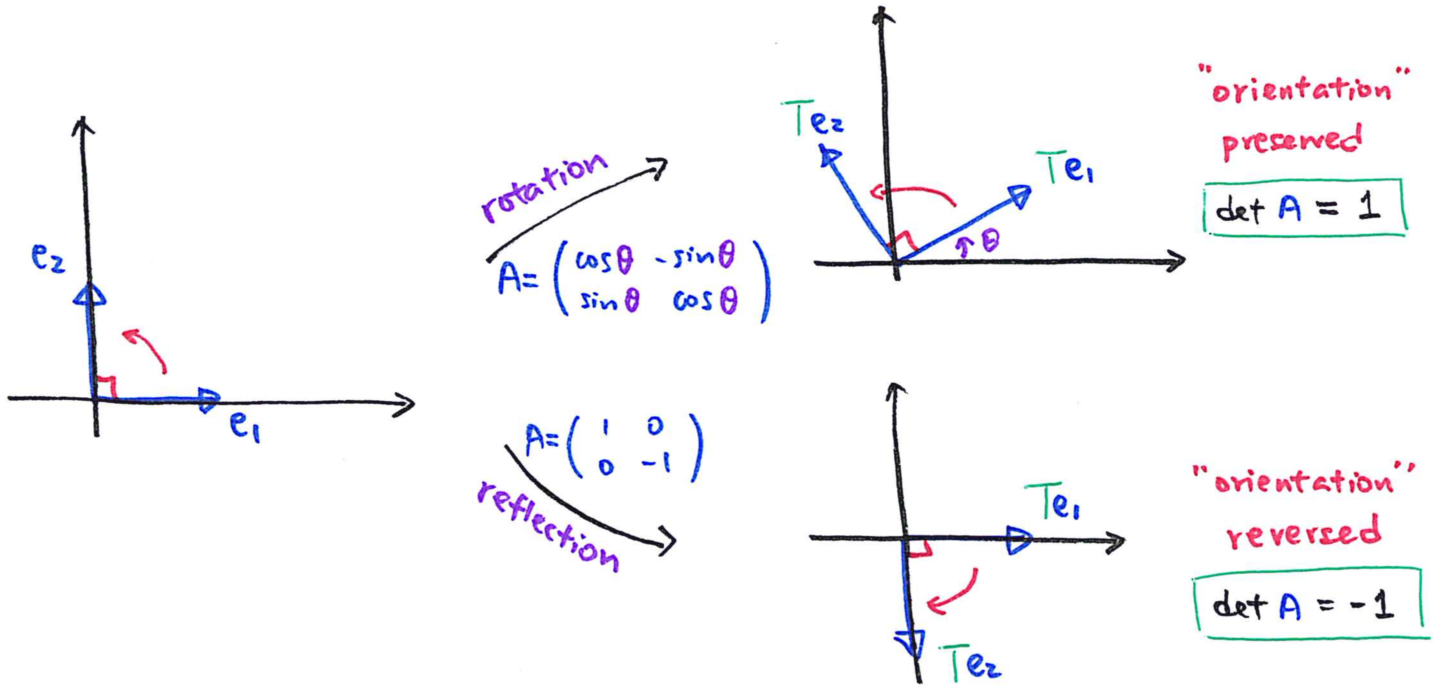
Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator preserving the standard inner product $\langle\, , \rangle$, i.e.

$$(*) \quad \boxed{\langle Tx, Ty \rangle = \langle x, y \rangle \quad \text{for } \underline{\underline{\text{all}}} \quad x, y \in \mathbb{R}^2}$$

- When $x = y$ in $(*)$ $\Rightarrow$ $\boxed{\|Tx\| = \|x\|}$ "length preserved"
- Since $\langle a, b \rangle = \|a\| \, \|b\| \cos\theta$, this and $(*)$ $\Rightarrow$ "angle preserved"

In particular, orthonormal basis $\xrightarrow{T}$ orthonormal basis

$$\underbrace{\{e_1, e_2\}}_{\text{"standard basis"}} \xmapsto{T} \{Te_1, Te_2\}$$



- Any composition of rotations and reflections still preserve length and angles. In fact, these are $\underline{\underline{\text{ALL}}}$ the tranformations in $\mathbb{R}^2$ which preserve length and angles!

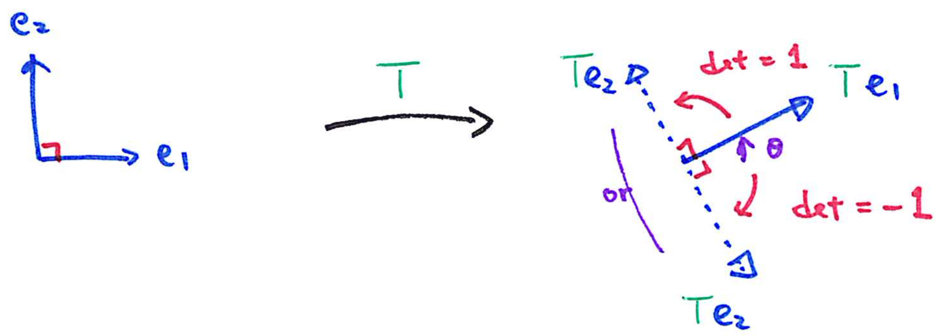
(i.e. satisfies (*))

Theorem: Any orthogonal operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ is either a rotation ($\det = 1$) or a reflection ($\det = -1$).

Proof: Since T preserves length, Te_1 is a unit vector, which can be obtained from e_1 by rotation of some angle θ.

Since T preserves angle, Te_2 must be a unit vector $\perp Te_1$.

There are only 2 possible choices:



$\square$

Since $\det(AB) = \det A \cdot \det B$, by considering the "parity":

$$\begin{cases} \text{rotation} \circ \text{rotation} = \text{rotation} & (1 \cdot 1 = 1) \\ \text{rotation} \circ \text{reflection} = \text{reflection} & (1 \cdot (-1) = -1) \\ \text{reflection} \circ \text{reflection} = \text{rotation} & ((-1) \cdot (-1) = 1) \end{cases}$$

Observe that the matrices of the rotations and reflections satisfy

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^t \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{A^t A = I = A A^t}$$

"orthogonal matrix"

Example: Show that $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{pmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{-1}{\sqrt5} \end{pmatrix}$$ represents a reflection.

Sol:

$$A^t A = \begin{pmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{-1}{\sqrt5} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{-1}{\sqrt5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A A^t$$

Hence, A is orthogonal $\Rightarrow$ ~~rotation~~ / reflection

but $\boxed{\det A = -1}$

Unitary / Orthogonal Operators & Matrices

Def$^{\underline{n}}$: (Operator form) Let $T: V \to V$ be a linear operator on a finite dim. inner product space $(V, \langle , \rangle)$.

T is unitary ($\mathbb{F} = \mathbb{C}$) / orthogonal ($\mathbb{F} = \mathbb{R}$) iff $\boxed{\|Tx\| = \|x\| \text{ for all } x \in V}$

(Matrix form) A matrix $A \in M_{n\times n}(\mathbb{F})$ is said to be unitary ($\mathbb{F} = \mathbb{C}$) / orthogonal ($\mathbb{F} = \mathbb{R}$) iff $\boxed{AA^* = I = A^*A}$

The Lemma below says that they are equivalent:

Lemma: If β is an orthonormal basis for $(V, \langle , \rangle)$, then

$$\boxed{\begin{array}{c} T: V \to V \\ \text{unitary / orthogonal} \end{array}} \iff \boxed{\begin{array}{c} [T]_\beta \text{ is} \\ \text{unitary / orthogonal} \end{array}}$$

operators   matrices

Proof: It follows from the Theorem below and $[T^*]_\beta = [T]_\beta^*$ for O.N.B. β.

Theorem: TFAE, for $T: V \to V$ on a finite dim. inner product space $(V, \langle , \rangle)$

(a) $\|Tx\| = \|x\|$ for all $x \in V$

(b) $TT^* = T^*T = I$.

(c) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in V$

(d) β O.N.B $\Rightarrow$ $T(\beta)$ O.N.B.

(e) there exist some O.N.B. β s.t. $T(\beta)$ is O.N.B.

Proof: $\boxed{(a) \Rightarrow (b)}$ We will need the following useful lemma:

"Useful Lemma": $\boxed{\langle x, Ux \rangle = 0 \text{ for all } x \in V}$ & $\boxed{U \text{ self adjoint}}$

$\Rightarrow$ $U = T_0$: zero transformation

Pf: Spectral Thm $\Rightarrow$ U diagonalizable and all eigenvalues $= 0$.

$Ux = \lambda x \Rightarrow \bar{\lambda} \|x\|^2 = \langle x, Ux \rangle = 0$.

By (a), we have for any $x \in V$

$$\langle x, x \rangle = \|x\|^2 = \|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle$$

i.e. $\langle x, \underbrace{(I - T^*T)}_{U} x \rangle = 0$ for all $x \in V$

U is self-adjoint: $(I - T^*T)^* = I^* - T^*T^{**} = I - T^*T$.

"Useful Lemma" $\Rightarrow$ $U = T_0$, i.e. $I = T^*T$. ($\Leftrightarrow$ $I = TT^*$)

$\boxed{\dim V < +\infty}$

$\boxed{(b) \Rightarrow (c)}$ For any $x, y \in V$.

$$\langle Tx, Ty \rangle = \langle x, T^*Ty \rangle \underset{(b)}{=} \langle x, y \rangle$$

$\boxed{(c) \Rightarrow (d) \Rightarrow (e)}$: trivial

$\boxed{(e) \Rightarrow (a)}$ Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an O.N.B. for V

such that $T(\beta) = \{Tv_1, Tv_2, \dots, Tv_n\}$ is still an O.N.B.

Let any $x \in V$, we can write

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \xRightarrow{\beta \text{ O.N.B.}} \|x\|^2 = \sum_{i=1}^{n} |a_i|^2$$

$$\|x\|^2 = \|Tx\|^2, \qquad \sum_{i=1}^{n} |a_i|^2 = \sum_{i=1}^{n} |a_i|^2$$

hence $Tx = a_1 Tv_1 + a_2 Tv_2 + \cdots + a_n Tv_n \xRightarrow{T(\beta) \text{ O.N.B}} \|Tx\|^2 = \sum_{i=1}^{n} |a_i|^2$

$\square$

Corollary: $\boxed{|\lambda| = 1}$ if $\lambda \in \mathbb{F}$ is an eigenvalue of a unitary / orthogonal operator.

Pf: $Tv = \lambda v \Rightarrow \|v\| = \|Tv\| = \|\lambda v\| = |\lambda| \cdot \|v\|$

$\square$

Corollary: When $\mathbb{F} = \mathbb{C}$, T is unitary

$$\Longleftrightarrow \begin{cases} \text{(i) } T \text{ is normal} \\ \text{(ii) } |\lambda| = 1 \text{ for } \underline{\underline{\text{all}}} \text{ eigenvalue of } T \end{cases}$$

Proof: "$\Rightarrow$" unitary $\Rightarrow$ normal, (ii) from previous corollary

"$\Leftarrow$" By Spectral Decomposition,

$$T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k. \quad \text{where } |\lambda_i| = 1.$$

$$\Rightarrow \quad T^* = \bar{\lambda}_1 T_1 + \bar{\lambda}_2 T_2 + \cdots + \bar{\lambda}_k T_k.$$

Hence,

$$TT^* = (\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k)(\bar{\lambda}_1 T_1 + \bar{\lambda}_2 T_2 + \cdots + \bar{\lambda}_k T_k)$$

$\boxed{T_i T_j = \begin{cases} T_0, & i \neq j \\ T_i, & i = j \end{cases}}$

$$= |\lambda_1|^2 T_1 + |\lambda_2|^2 T_2 + \cdots + |\lambda_k|^2 T_k$$

$$= T_1 + T_2 + \cdots + T_k = I$$

$\square$