MATH 2040A Linear Algebra II 2015-16

Week 13: Jordan canonical forms (textbook § 7.1, 7.2)

Diagonalization revisited once again

Recall that we have a characterization of the diagonalizability of a linear operator $T: V \to V$ on a finite dim. vector space V (over $\mathbb{F}$):

Theorem: $T: V \to V$ is diagonalizable

if and only if (1) the char. poly. splits over $\mathbb{F}$

$$f(\lambda) = (-1)^n (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}$$

(2) the eigenspaces have "maximal dimension":

i.e $\boxed{\dim E_{\lambda_i} = m_i}$ for all $i = 1, \ldots, k$.

FROM NOW ON, WE ASSUME $\mathbb{F} = \mathbb{C}$.

$\Rightarrow$ (1) is always satisfied!

Hence, "diagonalizable" $\Longleftrightarrow$ eigenspaces are "big enough".

Remember the following example of a "non-diagonalizable" matrix due to "small" eigenspaces:

$$\boxed{A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \Rightarrow \begin{cases} \text{only 1 eigenvalue } \boxed{\lambda = 1} \\ \dim E_1 = 1 < 2. \end{cases}$$

What else can we do if a matrix is not diagonalizable?

Schur's Lemma $\Rightarrow$ we can always make a matrix "upper triangular" (when $\mathbb{F} = \mathbb{C}$)

$$A \sim \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix}$$

Question: Can we do better?

Ans: Yes, "Jordan canonical form"

Theorem: Any $A \in M_{n\times n}(\mathbb{C})$ is similar to a matrix of the following form:

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} \lambda_N & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_N \end{matrix}} \end{pmatrix}$$

"Jordan canonical form"

where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of A (not nec. distinct!)

In other words, the "worst-case-scenario" for a matrix which is NOT diagonalizable is just having some 1's above the diagonals.

Examples of Jordan canonical forms:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix}$$

diagonal!

Thus, a Jordan canonical form consists of a certain number of "Jordan blocks" $\square$ along the "diagonals":

$$A = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & \lambda \end{pmatrix} \in M_{k\times k}(\mathbb{C})$$

"Jordan block" of size k with eigenvalue λ

We first study some basis properties of a Jordan block.

<u>Prop</u>: (1) A has only 1 eigenvalue λ. (multiplicity $= k$)

(2) $\boxed{\dim E_\lambda = 1}$ ($\Rightarrow$ A is <u>NOT</u> diagonalizable unless $k = 1$)

* (3) The smallest positive integer p s.t.

$$(A - \lambda I)^p = 0$$

is equal to its dimension k.

$$\left(\Rightarrow \boxed{N(A - \lambda I)^k = \mathbb{C}^k} \right)$$

* (4) If $\{e_1, \dots, e_k\}$ is the standard basis for $\mathbb{C}^k$, then $\boxed{(A - \lambda I)^i e_i} = 0$ for each $i = 1, \dots, k$.

<u>Proof</u>: (1) A upper triangular $\Rightarrow$ char. poly. $= f(t) = (-1)^k (t - \lambda)^k$.

Hence, λ is the only eigenvalue with multiplicity k.

(2)

$$A - \lambda I = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & 0 \end{pmatrix} \Rightarrow \text{null space} = \text{span}\{e_1\} = E_\lambda$$

$\dim = 1$.

(3) The 1's "marches up" one line at a time when we take powers of $A - \lambda I$

e.g.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\square$

By (4), only e_1 is an eigenvector of A, since $(A-\lambda I)e_1 = 0$.

Other e_i's are only annihilated by higher powers of $(A-\lambda I)$!

This motivates the following definition:

Def$^{\underline{n}}$: Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{C})$.

$x \in \mathbb{C}^n$ is a generalized eigenvector of A corresponding to the eigenvalue λ if (i) $x \neq 0$

and (ii) $\boxed{(A-\lambda I)^p x = 0 \text{ for some positive integer } p}$

We denote the generalized eigenspace by

$$\boxed{K_\lambda = \{ x \in \mathbb{C}^n \mid (A-\lambda I)^p x = 0 \text{ for some } p \geq 1 \}}$$

Ex: check K_λ is a subspace.

Remark: • Eigenvectors are generalized eigenvectors with $p = 1$.

• For a Jordan block as above, $K_\lambda = \mathbb{C}^k$.

Given a matrix $A \in M_{n\times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_k$ (distinct),

A diagonalizable $\iff$ $\boxed{\mathbb{C}^n = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}}$

In general, if A is NOT diagonalizable, we have

$$\boxed{\mathbb{C}^n \supsetneqq E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}}$$

i.e. some E_{λ_i} is too "small".

BUT, if we replace E_λ by K_λ, we always have

$$\boxed{\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}}$$

Main Theorem (Jordan Decomposition Theorem):

Let $A \in M_{n\times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_k$ (distinct) with corresponding multiplicities $m_1, \dots, m_k$. Then.

(1) $\boxed{\dim K_{\lambda_i} = m_i}$ for each $i = 1, \dots, k$

(2) $\boxed{\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}}$

(3) Each K_{λ_i} has a basis $\beta_i = \gamma_{1,i} \cup \cdots \cup \gamma_{\ell,i}$ where every $\gamma_{m,i}$ is a cycle: i.e.

p = length

$$\gamma_{m,i} = \{ \underbrace{(A - \lambda_i I)^{p-1} x}_{\text{initial vector}}, (A - \lambda_i I)^{p-2} x, \dots, \underbrace{x}_{\text{end vector}} \}$$

(an eigenvector!)

__Proof__: Postponed until later!

__Example__:

$$A = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \quad \text{"3×3 Jordan block".}$$

In this case, $K_\lambda = \mathbb{C}^3$ and we have a basis consisting of 1 cycle:

$$\gamma = \{ \underbrace{(A - \lambda I)^2 e_3}_{= e_1}, \underbrace{(A - \lambda I) e_3}_{= e_2}, e_3 \}.$$

Now, let us first address the "computational aspect" of Jordan decomposition:

__Question__: Given $A \in M_{n\times n}(\mathbb{C})$, how to find an invertible matrix Q s.t.

$$Q^{-1} A Q = J$$

Jordan canonical form

?

Example:

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$$

Step 1: Compute eigenvalues.

$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 2 & 3 \\ 1 & 3-\lambda & 3 \\ -1 & -2 & -2-\lambda \end{pmatrix} = -(\lambda - 1)^3 .$$

$\Rightarrow$ Only 1 eigenvalue $\boxed{\lambda = 1}$ with $\boxed{\text{multiplicity} = 3}$.

Step 2 : Compute eigenspaces.

$$E_1 = N(A - 1 \cdot I) = N \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Therefore, $\boxed{\dim E_1 = 2 < 3}$ ($\Rightarrow$ A NOT diagonalizable)

Step 3 : Determine the Jordan canonical form.

Since A has only one eigenvalue $\lambda = 1$ and is NOT diagonalizable, we are left with only two possibilities for its Jordan canonical form:

$$J = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \boxed{1} & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

impossible since each block □ would have exactly 1 eigenvector.

This is it! (up to order of blocks)

Step 4 : Find basis of K_λ consisting of cycles.

From Step 3, we need a basis

$$\beta = \gamma_1 \cup \gamma_2 = \{v_1\} \cup \{(A - \lambda I)v_2, v_2\}$$

eigenvectors

generalized eigenvector

We find the generalized eigenvector v_2 first :

Need $v_2 \in N(A-\lambda I)^2$ but $\underbrace{v_2 \notin N(A-\lambda I)}$

otherwise, cannot generate a cycle of length 2.

$$(A-1\cdot I)^2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 .$$

Therefore, we can pick any $v_2 \notin E_1$, e.g. $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\Rightarrow \quad \gamma_2 = \{ (A-1\cdot I)v_2 , v_2 \} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} .$$

In the end, take any $v_1 \in E_1$ s.t. v_1 is not parallel to $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$\beta = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{ie take } Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad Q^{-1} A Q = \begin{pmatrix} \boxed{1} & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} = \text{Jordan canonical form of } A$$

<u>One more example</u>:

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

<u>Step 1</u>: Compute eigenvalues.

A upper triangular $\Rightarrow$ only one eigenvalue

$\boxed{\lambda = -1}$ and $\boxed{m = 3}$

<u>Step 2</u>: Compute eigenspaces.

$$E_{-1} = N(A-(-1)\cdot I) = N\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} .$$

$\Rightarrow \boxed{\dim E_{-1} = 1 < 3}$ (A NOT diagonalizable)

Step 3: Determine Jordan canonical form.

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

can only have 1 block since $\dim E_{-1} = 1$

Step 4: Find basis of K_2 consisting of cycles.

From Step 3, we need a basis

$$\beta = \underset{\substack{\uparrow \\ \text{cycle of} \\ \text{length 3}}}{\gamma} = \{ (A+I)^2 v, (A+I)v, v \}$$

Need $v \in \underbrace{N(A+I)^3}_{= \mathbb{C}^3}$ but $v \notin N(A+I)$ or $N(A+I)^2$.

Note: $A+I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A+I) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

$(A+I)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A+I)^2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

We can take $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then

$$\beta = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \text{ ie } Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $Q^{-1} A Q = \begin{pmatrix} \boxed{\begin{matrix} -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{matrix}} \end{pmatrix}$ = Jordan canonical form of A

$\blacksquare$

Let's move on to some 4×4 examples!

Example:

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$$

Step 1: Compute eigenvalues.

The characteristic polynomial is:

$$f(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} 2-\lambda & -1 & 0 & 1 \\ 0 & 3-\lambda & -1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & -1 & 0 & 3-\lambda \end{pmatrix}$$

expand along 1st column since it has most 0's.

$$= (2-\lambda) \det \begin{pmatrix} 3-\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix}$$

expand along 3rd column remember $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$= (2-\lambda)(3-\lambda)\Big[\underbrace{(3-\lambda)(1-\lambda)+1}_{\lambda^2 - 4\lambda + 4 = (\lambda-2)^2}\Big]$$

$$= (\lambda-2)^3(\lambda-3)$$

Set $f(\lambda) = 0$, we get two eigenvalues: $\boxed{\begin{matrix} \lambda_1 = 2, & m_1 = 3 \\ \lambda_2 = 3, & m_2 = 1 \end{matrix}}$

Step 2: Compute eigenspaces.

$\boxed{\lambda_1 = 2}$: $E_2 = N(A - 2I) = N \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$\boxed{\dim E_2 = 2 < 3}$

$\boxed{\lambda_2 = 3}$: $E_3 = N(A - 3I) = N \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\boxed{\dim E_3 = 1 = 1}$

Note: A is NOT diagonalizable.

Step 3: Determine the Jordan canonical form

We know that

$$\boxed{\begin{array}{ll} \lambda_1 = 2, & \dim E_2 = 2 < 3 = m_1 \\ \lambda_2 = 3. & \dim E_3 = 1 = 1 = m_2 \end{array}}$$

From the multiplicities, we only have the following possibilities:

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & \\ & \boxed{3} \end{pmatrix} \quad \text{or} \quad \boxed{\begin{pmatrix} \boxed{2} & & \\ & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & & \boxed{3} \end{pmatrix}} \quad \text{or} \quad \begin{pmatrix} \boxed{2} & & & \\ & \boxed{2} & & \\ & & \boxed{2} & \\ & & & \boxed{3} \end{pmatrix}$$

impossible $\because$ $\dim E_2 = 1$ here.

This is it!

impossible $\because$ A NOT diagonalizable.

Step 4: Find basis of K_λ's consisting of cycles.

$K_3 = E_3$ $\quad \gamma = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ forms a basis.

$K_2 \neq E_2$ $\quad \gamma_1 \cup \gamma_2 = \{ v_1 \} \cup \{ (A - 2I) v_2, v_2 \}$

eigenvectors

gen. eigenvector s.t. $(A - 2I) v_2 \neq 0$.

Goal: Find $v_2 \in N(A - 2I)^2$ but $v_2 \notin N(A - 2I)$

$$N(A - 2I) = E_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

contains.

$$N(A - 2I)^2 = N \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

Therefore, one may take $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow (A - 2I) v_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

and take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in E_2$ but indep. of this!

Hence, if we choose the basis

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \text{i.e. } Q = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

then

$$Q^{-1} A Q = \begin{pmatrix} \boxed{2} & & & \\ & 2 & 1 & \\ & & 2 & \\ & & & \boxed{3} \end{pmatrix} = J$$

$\blacksquare$

<u>One more 4×4 example</u>:

$$A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}$$

<u>Step 1</u>: Compute eigenvalues.

$f(\lambda) = (\lambda - 2)^2 (\lambda - 4)^2 \leftarrow$ Exercise: check this.

$\Rightarrow$ There are 2 eigenvalues: $\boxed{\begin{matrix} \lambda_1 = 2, & m_1 = 2 \\ \lambda_2 = 4, & m_2 = 2 \end{matrix}}$

<u>Step 2</u>: Compute eigenspaces.

$\boxed{\lambda_1 = 2}$ $\quad E_2 = N(A - 2I) = N \begin{pmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$

$\boxed{\dim E_2 = 2 = m_1}$

$\boxed{\lambda_2 = 4}$ $\quad E_4 = N(A - 4I) = N \begin{pmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\boxed{\dim E_4 = 1 < 2 = m_2}$

$\hookrightarrow$ A not diagonalizable.

<u>Step 3</u>: Determine Jordan form.

$$J = \begin{pmatrix} \boxed{2} & & & \\ & \boxed{2} & & \\ & & 4 & 1 \\ & & & 4 \end{pmatrix}.$$

<u>Step 4</u> : Find basis of K_λ's consisting of cycles.

$K_2 = E_2 \;\Rightarrow\; \left\{ \begin{pmatrix} 2\\1\\0\\2 \end{pmatrix}, \begin{pmatrix} 0\\1\\2\\0 \end{pmatrix} \right\}$ is a desired basis

$K_4 \neq E_4 \;\Rightarrow\; \{ (A-4I)v ,\; v \}$ is a desired basis

We need $v \in N(A-4I)^2$ but $v \notin N(A-4I) = E_4$

$$N(A-4I)^2 = N\begin{pmatrix} 4 & 8 & -4 & -4 \\ 4 & 4 & 0 & -4 \\ 4 & 0 & 4 & -4 \\ 4 & 8 & -4 & -4 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} -1\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix} \right\}$$

Take $v = \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix}$, then $(A-4I)v = \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix}$.

As a result, if we choose the basis

$$\beta = \left\{ \begin{pmatrix} 2\\1\\0\\2 \end{pmatrix}, \begin{pmatrix} 0\\1\\2\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix} \right\} \quad , \text{ i.e. } \quad Q = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

then $$Q^{-1} A Q = \begin{pmatrix} \boxed{2} & & & \\ & \boxed{2} & & \\ & & 4 & 1 \\ & & & 4 \end{pmatrix} = J .$$

<u>One more 4x4 example</u>:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Char. poly $= f(\lambda) = \lambda^4 \;\Longrightarrow\;$ Eigenvalue : $\boxed{\lambda = 0 \;,\; m = 4}$

Eigenspace : $E_0 = N(A) = \operatorname{span}\left\{ \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} \right\}$ $\quad \boxed{\dim E_0 = 2}$

Jordan form : (2 possibilities) ← since $4 = 2+2 = 1+3$.

$$J = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \boxed{0} & & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

<u>Q</u>: How to tell which is the correct one?

Look at powers of the matrix:

$$J = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}} & \\ & \boxed{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}} \end{pmatrix} \Rightarrow J^2 = 0$$

$$\boxed{J = \begin{pmatrix} \boxed{0} & \\ & \boxed{\begin{matrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{matrix}} \end{pmatrix}} \Rightarrow J^2 = \begin{pmatrix} \boxed{0} & \\ & \boxed{\begin{matrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{matrix}} \end{pmatrix} \neq 0$$

we are in this case.

Since $Q^{-1}A^2Q = J^2$, and we see that

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

Need to find at last a basis of the form

$$\beta = \{v_1\} \cup \{A^2 v_2, A v_2, v_2\}$$

Find v_2 : $N(A^3) = \mathbb{C}^4$

$$N(A^2) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow \text{can choose } v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, $\beta = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, i.e. $Q = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

then $$Q^{-1}AQ = \begin{pmatrix} \boxed{0} & \\ & \boxed{\begin{matrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{matrix}} \end{pmatrix} = J$$

∎