THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050C Mathematical Analysis I

Tutorial 9 (March 20)

The following were discussed in the tutorial this week:

1 Limit Theorems

1.1 Squeeze Theorem. *Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$, and let c be a cluster point of A. If*

$$f(x) \le g(x) \le h(x) \quad \text{for all } \; x \in A, x \neq c,$$

and if $\lim_{x\to c} f = L = \lim_{x\to c} h$, then $\lim_{x\to c} g = L$.

Example 1.1. *(a)* $\lim_{x\to 0} \sin x = 0$

(b) $\lim_{x\to 0} \cos x = 1$

(c) $\lim_{x\to 0} \left(\frac{\cos x - 1}{x} \right) = 0$

(d) $\lim_{x\to 0} \left(\frac{\sin x}{x} \right) = 1$

2 Continuous Functions

Definition 2.1. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is continuous at c if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.*

Example 2.1 (Thomae's function)**.** *Let $A := \{x \in \mathbb{R} : x > 0\}$. Let $h : A \to \mathbb{R}$ be a function defined by*

$$h(x) = \begin{cases} 0 & \text{if } x \in A \setminus \mathbb{Q}, \\ 1/n & \text{if } x = m/n \in A \cap \mathbb{Q}, \text{ where } m, n \in \mathbb{N} \text{ with } \gcd(m, n) = 1. \end{cases}$$

Then h is continuous at every irrational point in A, but discontinuous at every rational point in A.

3 Classwork

1. Suppose f is a non-negative, real-valued function on $\mathbb{R}$ such that $\lim_{x\to x_0} f(x) = \ell$, where $x_0 \in \mathbb{R}$ and $\ell \in \mathbb{R}$, $\ell \ge 0$. Show that $\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{\ell}$.

2. Determine the limit $\lim_{x\to 0} \frac{1}{x} \sin(x^2)$.