THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050C Mathematical Analysis I

Tutorial 7 (March 6)

The following were discussed in the tutorial this week:

1 Contractive Sequences

Definition 1.1. *We say that a sequence (x_n) of real numbers is **contractive** if there exists a constant C, $0 < C < 1$, such that*

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#}$$

*The number C is called the **constant** of the contractive sequence.*

Remark. *Do not confuse (#) with the following condition:*

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#\#}$$

For example, $(\sqrt{n})$ satisfies (##) but it is not contractive.

1.1 Theorem. *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Example 1.1. *(Sequence of Fibonacci Fractions) Consider the sequence of Fibonacci fractions $x_n := f_n/f_{n+1}$, where (f_n) is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+2} := f_{n+1} + f_n$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $1/\varphi$, where $\varphi := (1 + \sqrt{5})/2$ is the Golden Ratio.*

2 Classwork

Let (x_n) be a sequence of real numbers defined by

$$\begin{cases} x_1 = 1, \quad x_2 = 2, \\ x_{n+2} := \dfrac{1}{3}(2x_{n+1} + x_n) \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

Show that (x_n) is convergent and find its limit.