MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 5

3.1-16

Note that for $n \geq 4$, $\frac{n}{n-1} \leq 2, \frac{1}{n-2} \leq \frac{2}{n}$ and $n! \geq n(n-1)(n-2)$. For arbitrary $\varepsilon > 0$, if $K > \max\{\frac{4}{\varepsilon}, 4\}$, then for all $n > K$,

$$\left|\frac{n^2}{n!}\right| \leq \frac{n^2}{n(n-1)(n-2)} \leq \frac{2}{n-2} \leq \frac{4}{n} < \varepsilon.$$

3.2-1(b)

Suppose that $a := \lim x_n$ exists. Take $\varepsilon = \frac{1}{3}$ so that there exists a natural number K satisfying

$$|a - x_n| < \frac{1}{3} \quad \forall n \geq K.$$

Notice that $\frac{1}{2} \leq \frac{n}{n+1} < 1, \forall n \in \mathbb{N}$. If n is an odd natural number with $n \geq K$ this gives $\left|a + \frac{n}{n+1}\right| < \frac{1}{3}$, implying $-2 < -\frac{n}{n+1} - \frac{1}{3} < a < -\frac{n}{n+1} + \frac{1}{3} < 0$, i.e. $-2 < a < 0$. If n is an even natural number with $n \geq K$ this gives $\left|a - \frac{n}{n+1}\right| < \frac{1}{3}$, implying $0 < \frac{n}{n+1} - \frac{1}{3} < a < \frac{n}{n+1} + \frac{1}{3} < 2$, i.e. $0 < a < 2$. Since a cannot satisfy both inequalities simultaneously, a contradiction. Hence the sequence is divergent.

3.2-1(d)

$x_n = \frac{2n^2+3}{n^2+1} = \frac{2+3/n^2}{1+1/n^2}$. Set $a_n = 2 + 3/n^2$ and $b_n = 1 + 1/n^2$. Since $\lim a_n = 2$ and $\lim b_n = 1$, apply Theorem 3.2.3(b) to $x_n = \frac{a_n}{b_n}$ and $\lim x_n = 2$.

3.2-5(b)

Suppose that $((-1)^n n^2)$ is convergent thus bounded. There exists some real number $M > 0$, so that $\left|(-1)^n n^2\right| < M, \forall n \in \mathbb{N}$. Take $N_0 \in \mathbb{N}$ satisfying $N_0 > M + 1$ by the Archimedean Property. Then $(N_0)^2 > (M+1)^2 > M$, contradiction.

3.2-12

$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{b+a\cdot a^n/b^n}{1+a^n/b^n}$. Set $x_n = b + a \cdot a^n/b^n$ and $y_n = 1 + a^n/b^n$. Since $\lim x_n = b$ and $\lim y_n = 1$, apply Theorem 3.2.3(b) to obtain $\lim \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = b$.

3.2-14(a)

(a) From the inequalities $1 \leq n$ and $n \leq n^n$, we have $1 \leq n^{1/n^2} \leq n^{1/n}, \forall n \in \mathbb{N}$. Since $\lim n^{1/n} = 1$, apply Theorem 3.2.7 and $\lim n^{1/n^2} = 1$.

(b) From the inequalities $1 \leq n!$ and $n! \leq n^n$, we have $1 \leq (n!)^{1/n^2} \leq n^{1/n}, \forall n \in \mathbb{N}$. Since $\lim n^{1/n} = 1$, apply Theorem 3.2.7 and $\lim (n!)^{1/n^2} = 1$.

3.2-18

Let r be a number so that $1 < r < L$ and let $\varepsilon = L - r$. There exists a number $K \in \mathbb{N}$ so that if $n \geq K$ then

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon$$

and

$$\frac{x_{n+1}}{x_n} > L - \varepsilon = r.$$

As $x_n > 0, \forall n \in \mathbb{N}$, $x_{n+K} > r x_{n+K-1} > \cdots > r^n x_K, \forall n \in \mathbb{N}$. Since $r > 1$, for any positive real number M, take n large enough satisfying $r^n > M/x_K$. Thus (x_n) is unbounded and divergent.