MATH2050A HW7 Solution

2. Let $\epsilon > 0$. Since $x \geq 1$ and $y \geq 1$,

$$|\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{(y+x)(y-x)}{x^2y^2}| = |\frac{1}{xy^2} + \frac{1}{x^2y}||y-x| \leq 2|y-x|.$$

Let $\delta = \frac{\epsilon}{2}$, then whenever $|y-x| < \delta$, $|f(x) - f(y)| \leq 2|y-x| < \epsilon$. Thus $f(x)$ is uniformly continuous on A.

Take $\epsilon = 1$, let $\delta > 0$ be given. There exists $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$. Take $x = \frac{1}{N} \in B$ and $y = \frac{1}{N+1} \in B$, $|x-y| = \frac{1}{N(N+1)} \leq \delta$. But $|f(x) - f(y)| = 2N+1 \geq \epsilon$. Thus $f(x)$ is not uniformly continuous on B.

6. Let $\epsilon > 0$. Since f and g are bounded function on A, there exists M_f, M_g s.t. $|f(x)| < M_f$ and $|g(x)| < M_g$ for any $x \in A$. Let $M = \max M_f, M_g$.

By uniformly continuity of f and g, there exists $\delta_f, \delta_g > 0$ s.t. if $|x-y| < \min\{\delta_f, \delta_g\}$, then $|f(x) - f(y)| < \frac{\epsilon}{2M}$ and $|g(x) - g(y)| < \frac{\epsilon}{2M}$. Let $\delta = \min\{\delta_f, \delta_g\}$. If $x, y \in A$ and $|x-y| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)||(g(x) - g(y)| + |f(x) - f(y)||g(y)| \\ &< M\frac{\epsilon}{2M} + \frac{\epsilon}{2M}M \\ &\leq \epsilon. \end{aligned}$$

Thus fg is uniform continuous on A.

7.
- f is uniformly continuous.
  Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. If $x, y \in \mathbb{R}$ and $|x-y| < \delta$, then $|f(x) - f(y)| = |x-y| < \epsilon$. Thus f is uniformly continuous.
- g is uniformly continuous.
  Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. If $x, y \in \mathbb{R}$ and $|x-y| < \delta$, by mean value theorem, there exists ζ between x and y, s.t.

  $$\frac{|g(x) - g(y)|}{|x-y|} = \cos(\zeta) \leq 1$$

  Thus $|g(x) - g(y)| \leq |x-y| < \delta = \epsilon$. g is uniformly continuous.
  Remark. Here you may also use the conclusion of Q.14 from the same exercise, which is better and doesn't involve any derivative results (not yet covered). Also, you may also google for some elementary proof using only sum-to-product.
- fg is not uniformly continuous.
  Take $\epsilon = 1$, let $\delta > 0$ be given. Now we consider $x = 2\pi N$ and $y = 2\pi(N + \delta/(4\pi))$, where $N > 0$ is to be determined. Note that $|x-y| = \delta/2$. Then

  $$|fg(x) - fg(y)| = 2\pi(N + \delta/(4\pi))|\sin(\delta/(4\pi))| > N|\sin(\delta/(4\pi))| > 1 = \epsilon,$$

  if we choose N large enough so that $N > 1/|\sin(\delta/(4\pi))|$. Thus fg is not uniformly continuous.