

MATH 2050A: Mathematical Analysis I (2017 1st term)

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1 Compact Sets in \mathbb{R}

Throughout this section, let (x_n) be a sequence in \mathbb{R} . Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, \dots\} \mapsto n_k \in \{1, 2, \dots\}$.

In this case, note that for each positive integer N , there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Let us first recall the following two important theorems in real line.

Theorem 1.1 Nested Intervals Theorem *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.*

$$(i) : I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

$$(ii) : \lim_n (b_n - a_n) = 0.$$

Then there is a unique real number ξ such that $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Proof: See [1, Theorem 2.5.2, Theorem 2.5.3]. □

Theorem 1.2 (Bolzano-Weierstrass Theorem) *Every bounded sequence in \mathbb{R} has a convergent subsequence.*

Proof: See [1, Theorem 3.4.8]. □

Definition 1.3 A subset A of \mathbb{R} is said to be *compact* (more precise, *sequentially compact*) if every sequence in A has a convergent subsequence with the limit in A .

We are now going to characterize the compact subsets of \mathbb{R} . The following is an important notation in mathematics.

Definition 1.4 A subset A is said to be *closed* in \mathbb{R} if it satisfies the condition:

if (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim x_n \in A$.

Example 1.5 (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}; \mathbb{N}$; the empty set \emptyset and \mathbb{R} all are closed subsets of \mathbb{R} .

(ii) (a, b) and \mathbb{Q} are not closed.

The following Proposition is one of the basic properties of a closed subset which can be directly shown by the definition. So, the proof is omitted here.

Proposition 1.6 *Let A be a subset of \mathbb{R} . The following statements are equivalent.*

(i) A is closed.

(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.

The following is an important characterization of a compact set in \mathbb{R} . **Warning:** this result is not true for the so-called *metric spaces* in general.

Theorem 1.7 *Let A be a closed subset of \mathbb{R} . Then the following statements are equivalent.*

(i) A is compact.

(ii) A is closed and bounded.

Proof: It is clear that the result follows if $A = \emptyset$. So, we assume that A is non-empty.

For showing (i) \Rightarrow (ii), assume that A is compact.

We first claim that A is closed. Let (x_n) be a sequence in A . Then by the compactness of A , there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. So, if (x_n) is convergent, then $\lim_n x_n = \lim_k x_{n_k} \in A$. Therefore, A is closed.

Next, we are going to show the boundedness of A . Suppose that A is not bounded. Fix an element $x_1 \in A$. Since A is not bounded, we can find an element $x_2 \in A$ such that $|x_2 - x_1| > 1$. Similarly, there is an element $x_3 \in A$ such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can obtain a sequence (x_n) in A such that $|x_n - x_m| > 1$ for $m \neq n$. From this, we see that the sequence (x_n) does not have a convergent subsequence. In fact, if (x_n) has a convergent subsequence (x_{n_k}) . Put $L := \lim_k x_{n_k}$. Then we can find a pair of sufficient large positive integers p and q with $p \neq q$ such that $|x_{n_p} - L| < 1/2$ and $|x_{n_q} - L| < 1/2$. This implies that $|x_{n_p} - x_{n_q}| < 1$. It leads to a contradiction because $|x_{n_p} - x_{n_q}| > 1$ by the choice of the sequence (x_n) . Thus, A is bounded.

It remains to show (ii) \Rightarrow (i). Suppose that A is closed and bounded.

Let (x_n) be a sequence in A . Thus, (x_n) . Then the Bolzano-Weierstrass Theorem assures that there is a convergent subsequence (x_{n_k}) . Then by the closeness of A , $\lim_k x_{n_k} \in A$. Thus A is compact.

The proof is finished.

□

2 Appendix: Compact sets in \mathbb{R} , Part 2

For convenience, we call a collection of open intervals $\{J_\alpha : \alpha \in \Lambda\}$ an *open intervals cover* of a given subset A of \mathbb{R} , where Λ is an arbitrary non-empty index set, if each J_α is an open

interval (not necessary bounded) and

$$A \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha.$$

Theorem 2.1 Heine-Borel Theorem: *Any closed and bounded interval $[a, b]$ satisfies the following condition:*

(HB) *Given any open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$, we can find finitely many $J_{\alpha_1}, \dots, J_{\alpha_N}$ such that $[a, b] \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N}$*

Proof: Suppose that $[a, b]$ does not satisfy the above Condition (HB). Then there is an open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$ but it has no finite sub-cover. Let $I_1 := [a_1, b_1] = [a, b]$ and m_1 the mid-point of $[a_1, b_1]$. Then by the assumption, $[a_1, m_1]$ or $[m_1, b_1]$ cannot be covered by finitely many J_α 's. We may assume that $[a_1, m_1]$ cannot be covered by finitely many J_α 's. Put $I_2 := [a_2, b_2] = [a_1, m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n, b_n]$ with the following properties:

- (a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$;
- (b) $\lim_n (b_n - a_n) = 0$;
- (c) each I_n cannot be covered by finitely many J_α 's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \leq \xi \leq b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is finished.

□

Remark 2.2 The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.

For example, notice that $\{J_n := (1/n, 1) : n = 1, 2, \dots\}$ is an open interval covers of $(0, 1)$ but you cannot find finitely many J_n 's to cover the open interval $(0, 1)$.

The following is a very important feature of a compact set.

Theorem 2.3 *Let A be a subset of \mathbb{R} . Then the following statements are equivalent.*

- (i) *For any open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of A , we can find finitely many $J_{\alpha_1}, \dots, J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N}$.*
- (ii) *A is compact.*
- (iii) *A is closed and bounded.*

Proof: The result will be shown by the following path

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).$$

For $(i) \Rightarrow (ii)$, assume that the condition (i) holds but A is not compact. Then there is a sequence (x_n) in A such that (x_n) has no subsequence which has the limit in A . Put $X = \{x_n : n = 1, 2, \dots\}$. Then X is infinite. Also, for each element $a \in A$, there is $\delta_a > 0$ such that $J_a := (a - \delta_a, a + \delta_a) \cap X$ is finite. Indeed, if there is an element $a \in A$ such that $(a - \delta, a + \delta) \cap A$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit a . On the other hand, we have $A \subseteq \bigcup_{a \in A} J_a$. Then by the compactness of A , we can find finitely many a_1, \dots, a_N such that $A \subseteq J_{a_1} \cup \dots \cup J_{a_N}$. So we have $X \subseteq J_{a_1} \cup \dots \cup J_{a_N}$. Then by the choice of J_a 's, X must be finite. This leads to a contradiction. Therefore, A must be compact.

The implication $(ii) \Rightarrow (iii)$ follows from Theorem 1.7 at once.

It remains to show $(iii) \Rightarrow (i)$. Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha \in \Lambda}$ be an open intervals cover of A . Notice that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since A is closed by using Proposition 1.6. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha \cup \bigcup_{x \in [a, b] \setminus A} I_x.$$

Using the Heine-Borel Theorem 2.1, we can find finitely many J_α 's and I_x 's, say $J_{\alpha_1}, \dots, J_{\alpha_N}$ and I_{x_1}, \dots, I_{x_K} , such that $[a, b] \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N} \cup I_{x_1} \cup \dots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of I_x . Therefore, we have $A \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N}$ and hence A is compact.

The proof is finished. □

Remark 2.4 In fact, the condition in Theorem 2.3(i) is the usual definition of a *compact set* for a general topological space. More precise, if a set A satisfies the Definition 1.4, then A is said to be *sequentially compact*. Theorem 2.3 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of \mathbb{R} . However, these two notation are different for a general topological space.

Strongly recommended: take the courses: MATH 3060; MATH3070 for the next step.

3 Continuous functions defined on compact sets

Throughout this section, let A be a non-empty subset of \mathbb{R} and $f : A \rightarrow \mathbb{R}$ a function defined on A .

Proposition 3.1 *Let f be a continuous function defined on a compact subset A of \mathbb{R} . Then $f(A)$ is a compact subset of \mathbb{R} .*

Proof: Method I: By using Theorem 2.3 $(i) \Leftrightarrow (iii)$, it suffices to show that $f(A)$ is a closed bounded subset of \mathbb{R} .

Claim 1: $f(A)$ is bounded.

Suppose not. Then for each positive integer n , there is an element $x_n \in A$ such that $|f(x_n)| > n$.

Since A is compact, there is a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in A$. This gives $\lim_k f(x_{n_k}) = f(a)$ because f is continuous on a and hence, $(f(x_{n_k}))$ is a bounded sequence. This leads to a contradiction to the choice of (x_n) which satisfies $|f(x_{n_k})| > n_k$ for all $k = 1, 2, \dots$

Claim 2: $f(A)$ is a closed subset of \mathbb{R} , that is, $y \in f(A)$ whenever, a sequence (x_n) in A satisfying $\lim_n f(x_n) = y$.

In fact, there is a convergent subsequence (x_k) with $z := \lim_k x_k \in A$ by using the compactness of A again. This gives $y = \lim_k f(x_{n_k}) = f(z) \in f(A)$ as desired since f is continuous on A .

Method II: Alternatively, we are going to use Theorem 2.3 (i) \Leftrightarrow (ii).

Let $\{J_i\}_{i \in I}$ be an open interval covers of $f(A)$. We may assume $J_i \cap f(A) \neq \emptyset$ for each $i \in I$. Notice that since J_i is an open interval and f is continuous, we see that if $f(x) \in J_i$, then we can find $\delta_x > 0$ such that $f(z) \in J_i$ whenever $z \in A$ with $|z - x| < \delta_x$. Notice that we have $A \subseteq \bigcup_{x \in A} J_x$, where $V_x := (x - \delta_x, x + \delta_x)$ and hence, $\{V_x : x \in A\}$ forms an open intervals cover of A . By using the equivalence (i) \Leftrightarrow (ii) in Theorem 2.3, we can find finitely many x_1, \dots, x_n in A such that $A \subseteq V_{x_1} \cup \dots \cup V_{x_n}$. For each $k = 1, \dots, n$, then $f(x_k) \in J_{i_k}$ for some $i_k \in I$. Now if $x \in A$, then $x \in V_{x_k}$ for some $k = 1, \dots, n$. This gives $f(x) \in J_{i_k}$ and thus, $f(A) \subseteq J_{i_1} \cup \dots \cup J_{i_n}$. The proof is finished. \square

Corollary 3.2 *If $f : A \rightarrow \mathbb{R}$ is a continuous injection and A is compact, then the inverse map $f^{-1} : f(A) \rightarrow A$ is also continuous.*

Proof: Let $B = f(A)$ and $g = f^{-1} : B \rightarrow A$. Suppose that g is not continuous at some $b \in B$. Put $a = g(b) \in A$. Then there are $\eta > 0$ and a sequence (y_n) in B such that $\lim y_n = b$ but $|g(y_n) - g(b)| \geq \eta$ for all n . Let $x_n := g(y_n) \in A$. So, by the compactness of A , there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} \in A$. Let $a' = \lim_k x_{n_k}$. Then we have $f(a') = \lim_k f(x_{n_k}) = \lim_k y_{n_k} = b$. On the other hand, since $|g(y_n) - g(b)| \geq \eta$ for all n , we see that

$$|x_{n_k} - a| = |g(y_{n_k}) - g(b)| \geq \eta > 0$$

for all k and hence $|a' - a| > 0$. This implies that $a \neq a'$ but $f(a') = b = f(a)$. It contradicts to f being injective.

The proof is finished. \square

Remark 3.3 The assumption of the compactness in the last assertion of Proposition 3.2 is essential. For example, consider $A = [0, 1) \cup [2, 3]$ and define $f : A \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ x - 1 & \text{if } x \in [2, 3]. \end{cases}$$

Then $f(A) = [0, 2]$ and f is a continuous bijection from A onto $[0, 2]$ but $f^{-1} : [0, 2] \rightarrow A$ is not continuous at $y = 1$.

Example 3.4 By Proposition 3.2, it is impossible to find a continuous surjection from $[0, 1]$ onto $(0, 1)$ since $[0, 1]$ is compact but $(0, 1)$ is not. Thus $[0, 1]$ is not homeomorphic to $(0, 1)$.

Proposition 3.5 *Suppose that f is continuous on A . If A is compact, then there are points c and b in A such that*

$$f(c) = \max\{f(x) : x \in A\} \text{ and } f(b) = \min\{f(x) : x \in A\}.$$

Proof: By considering the function $-f$ on A , it needs to show that $f(c) = \max\{f(x) : x \in A\}$ for some $c \in A$.

Method I:

We first claim that f is bounded on A , that is, there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Suppose not. Then for each $n \in \mathbb{N}$, we can find $a_n \in A$ such that $|f(a_n)| > n$. Recall that A is compact if and only if it is closed and bounded (see Theorem ??). So, (a_n) is a bounded sequence in A . Then by the Bolzano-Weierstrass Theorem, there is a convergent subsequence (a_{n_k}) of (a_n) . Put $a = \lim_k a_{n_k}$. Since A is closed and f is continuous, $a \in A$, from this, it follows that $f(a) = \lim_k f(a_{n_k})$. It is absurd because $n_k < |f(a_{n_k})| \rightarrow |f(a)|$ for all k and $n_k \rightarrow \infty$. So f must be bounded. So $L := \sup\{f(x) : x \in A\}$ must exist by the Axiom of Completeness.

It remains to show that there is a point $c \in A$ such that $f(c) = L$. In fact, by the definition of supremum, there is a sequence (x_n) in A such that $\lim_n f(x_n) = L$. Then by the Bolzano-Weierstrass Theorem again, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. If we put $c := \lim_k x_{n_k} \in A$, then $f(c) = \lim_k f(x_{n_k}) = L$ as desired. The proof is finished.

Method II:

We first claim that f is bounded above. Notice that for each $x \in A$, there is $\delta_x > 0$ such that $f(y) < f(x) + 1$ whenever $y \in A$ with $|x - y| < \delta_x$ since f is continuous on A . Now if we put $J_x := (x - \delta_x, x + \delta_x)$ for each $x \in A$, then $A \subseteq \bigcup_{x \in A} J_x$. So, by the compactness of A , we can find finitely many x_1, \dots, x_N in A such that $A \subseteq J_{x_1} \cup \dots \cup J_{x_N}$ and it follows that for each $x \in A$, we have $f(x) < 1 + f(x_k)$ for some $k = 1, \dots, N$. Now if we put $M := \max\{1 + f(x_1), \dots, 1 + f(x_N)\}$, then f is bounded above by M on A .

Put $L := \sup\{f(x) : x \in A\}$. It remains to show that there is an element $c \in A$ such that $f(c) = L$. Suppose not. Notice that since $f(x) \leq L$ for all $x \in A$, we have $f(x) < L$ for all $x \in A$ under this assumption. Therefore, by the continuity of f , for each $x \in A$, there are $\varepsilon_x > 0$ and $\eta_x > 0$ such that $f(y) < f(x) + \varepsilon_x < L$ whenever $y \in A$ with $|y - x| < \delta_x$. Put $I_x := (x - \eta_x, x + \eta_x)$. Then $A \subseteq \bigcup_{x \in A} I_x$. By the compactness of A again, A can be covered by finitely many I_{x_1}, \dots, I_{x_N} . If we let $L' := \max\{f(x_1) + \varepsilon_{x_1}, \dots, f(x_N) + \varepsilon_{x_N}\}$, then $f(x) < L' < L$ for all $x \in A$. It contradicts to L being the least upper bound for the set $\{f(x) : x \in A\}$. The proof is complete. \square

Definition 3.6 We say that a function f is *upper semi-continuous* (resp. *lower semi-continuous*) on A if for each element $z \in A$ and for any $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) < f(z) + \varepsilon$ (resp. $f(z) - \varepsilon < f(x)$) whenever $x \in A$ with $|x - z| < \delta$.

Remark 3.7 (i) It is clear that a function is continuous if and only if it is upper semi-continuous and lower semi-continuous. However, an upper semi-continuous function need not be continuous. For example, define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

(ii) From the **Method II** above, we see that if f is upper semi-continuous (resp. lower semi-continuous) on a compact set A , then the function f attains the supremum (resp. infimum) on A .

4 Uniform Continuous Functions

Definition 4.1 A function $f : A \rightarrow \mathbb{R}$ is said to be uniformly continuous on A if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$.

Remark 4.2 It is clear that if f is uniformly continuous on A , then it must be continuous on A . However, the converse does not hold. For example, consider the function $f : (0, 1] \rightarrow \mathbb{R}$ defined by $f(x) := 1/x$. Then f is continuous on $(0, 1]$ but it is not uniformly continuous on $(0, 1]$. Notice that f is not uniformly continuous on A means that

there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Notice that $1/x \rightarrow \infty$ as $x \rightarrow 0+$. So if we let $\varepsilon = 1$, then for any $\delta > 0$, we choose $n \in \mathbb{N}$ such that $1/n < \delta$ and thus we have $|1/2n - 1/n| = 1/2n < \delta$ but $|f(1/n) - f(1/2n)| = n > 1 = \varepsilon$. Therefore, f is not uniformly continuous on $(0, 1]$.

Example 4.3 Let $0 < a < 1$. Define $f(x) = 1/x$ for $x \in [a, 1]$. Then f is uniformly continuous on $[a, 1]$. In fact for $x, y \in [a, 1]$, we have

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} \leq \frac{|x - y|}{a^2}.$$

So for any $\varepsilon > 0$, we can take $0 < \delta < a^2\varepsilon$. Thus if $x, y \in [a, 1]$ with $|x - y| < \delta$, then we have $|f(x) - f(y)| < \varepsilon$ and hence f is uniformly continuous on $[a, 1]$.

Proposition 4.4 *If f is continuous on a compact set A , then f is uniformly continuous on A .*

Proof: Compactness argument:

Let $\varepsilon > 0$. Since f is continuous on A , then for each $x \in A$, there is $\delta_x > 0$, such that $|f(y) - f(x)| < \varepsilon$ whenever $y \in A$ with $|y - x| < \delta_x$. Now for each $x \in A$, set $J_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Then $A \subseteq \bigcup_{x \in A} J_x$. By the compactness of A , there are finitely many $x_1, \dots, x_N \in A$ such that $A \subseteq J_{x_1} \cup \dots \cup J_{x_N}$. Now take $0 < \delta < \min(\frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_N}}{2})$. Now for $x, y \in A$ with $|x - y| < \delta$, then $x \in J_{x_k}$ for some $k = 1, \dots, N$, from this it follows that $|x - x_k| < \frac{\delta_{x_k}}{2}$ and $|y - x_k| \leq |y - x| + |x - x_k| < \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \delta_{x_k}$. So for the choice of δ_{x_k} , we have $|f(y) - f(x_k)| < \varepsilon$ and $|f(x) - f(x_k)| < \varepsilon$. Thus we have shown that $|f(x) - f(y)| < 2\varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$. The proof is finished.

Sequentially compactness argument:

Suppose that f is not uniformly continuous on A . Then there is $\varepsilon > 0$ such that for each $n = 1, 2, \dots$, we can find x_n and y_n in A with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Notice that by the sequentially compactness of A , (x_n) has a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in A$. Now applying sequentially compactness of A for the sequence (y_{n_k}) , then (y_{n_k}) contains a convergent subsequence $(y_{n_{k_j}})$ such that $b := \lim_j y_{n_{k_j}} \in A$. On the other hand, we also have $\lim_j x_{n_{k_j}} = a$. Since $|x_{n_{k_j}} - y_{n_{k_j}}| < 1/n_{k_j}$ for all j , we see that $a = b$. This implies that $\lim_j f(x_{n_{k_j}}) = f(a) = f(b) = \lim_j f(y_{n_{k_j}})$. This leads to a contradiction since we always have $|f(x_{n_{k_j}}) - f(y_{n_{k_j}})| \geq \varepsilon > 0$ for all j by the choice of x_n and y_n above. The proof is finished. \square

Proposition 4.5 *Let f be a continuous function defined on a bounded subset A of \mathbb{R} . Then the following statements are equivalent.*

(i): f is uniformly continuous on A .

(ii): There is a unique continuous function F defined on the closure \overline{A} such that $F(x) = f(x)$ for all $x \in A$.

Proof: Notice that since A is bounded then so is \overline{A} . This implies that \overline{A} is compact. The Part (ii) \Rightarrow (i) follows Proposition 4.4 at once.

The proof of Part (i) \Rightarrow (ii) is divided by the following assertions. Assume that f is uniformly continuous on A .

Claim 1. If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim f(x_n)$ exists.

It needs to show that $(f(x_n))$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. Then by the uniform continuity of f on A , there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$. Notice that (x_n) is a Cauchy sequence since it is convergent. Thus, there is a positive integer N such that $|x_m - x_n| < \delta$ for all $m, n \geq N$. This implies that $|f(x_m) - f(x_n)| < \varepsilon$ for all $m, n \geq N$ and hence, **Claim 1** follows.

Claim 2. If (x_n) and (y_n) both are convergent sequences in A and $\lim x_n = \lim y_n$, then $\lim f(x_n) = \lim f(y_n)$.

By **Claim 1**, $L := \lim f(x_n)$ and $L' = \lim f(y_n)$ both exist. For any $\varepsilon > 0$, let $\delta > 0$ be found as in **Claim 1**. Since $\lim x_n = \lim y_n$, there is $N \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq N$ and hence, we have $|f(x_n) - f(y_n)| < \varepsilon$ for all $n \geq N$. Taking $n \rightarrow \infty$, we see that $|L - L'| \leq \varepsilon$ for all $\varepsilon > 0$. So $L = L'$. **Claim 2** follows.

Recall that an element $x \in \overline{A}$ if and only if there is a sequence (x_n) in A converging to x .

Now for each $x \in \overline{A}$, we define

$$F(x) := \lim f(x_n)$$

if (x_n) is a sequence in A with $\lim x_n = x$. It follows from **Claim 1** and **Claim 2** that F is a well defined function defined on \overline{A} and $F(x) = f(x)$ for all $x \in A$.

So, it remains to show that F is continuous. Then F is a continuous extension of f to \overline{A} as desired.

Now suppose that F is not continuous at some point $z \in \overline{A}$. Then there is $\varepsilon > 0$ such that for any $\delta > 0$, there is $x \in \overline{A}$ satisfying $|x - z| < \delta$ but $|F(x) - F(z)| \geq \varepsilon$. Notice that for any $\delta > 0$ and if $|x - z| < \delta$ for some $x \in \overline{A}$, then we can choose a sequence (x_i) in A such that $\lim x_i = x$. Therefore, we have $|x_i - z| < \delta$ and $|f(x_i) - F(z)| \geq \varepsilon/2$ for any i large enough. Therefore, for any $\delta > 0$, we can find an element $x \in A$ with $|x - z| < \delta$ but $|f(x) - F(z)| \geq \varepsilon/2$. Now consider $\delta = 1/n$ for $n = 1, 2, \dots$. This yields a sequence (x_n) in A which converges to z but $|f(x_n) - F(z)| \geq \varepsilon/2$ for all n . However, we have $\lim f(x_n) = F(z)$ by the definition of F which leads to a contradiction. Thus F is continuous on \overline{A} .

Finally the uniqueness of such continuous extension is clear.

The proof is finished. □

Example 4.6 By using Proposition 4.5, the function $f(x) := \sin \frac{1}{x}$ defined on $(0, 1]$ cannot be continuously extended to the set $[0, 1]$.

Definition 4.7 Let A be a non-empty subset of \mathbb{R} . A function $f : A \rightarrow \mathbb{R}$ is called a Lipschitz if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. In this case. Furthermore, if we can find such $0 < C < 1$, then we call f a contraction.

It is clear that we have the following property.

Proposition 4.8 *Every Lipschitz function is uniformly continuous on its domain.*

Example 4.9 (i) : The sine function $f(x) = \sin x$ is a Lipschitz function on \mathbb{R} since we always have $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$ (by using the equation $\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$ and the fact $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.)

(ii) : Define a function f on $[0, 1]$ by $f(x) = x \sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$. Then f is continuous on $[0, 1]$ and thus f is uniformly continuous on $[0, 1]$. But notice that f is not a Lipschitz function. In fact, for any $C > 0$, if we consider $x_n = \frac{1}{2n\pi + (\pi/2)}$ and $y_n = \frac{1}{2n\pi}$, then $|f(x_n) - f(y_n)| > C|x_n - y_n|$ if and only if

$$\frac{2}{\pi} \cdot \frac{(2n\pi + \frac{\pi}{2})(2n\pi)}{2n\pi + \frac{\pi}{2}} = 4n > C.$$

Therefore, for any $C > 0$, there are $x, y \in [0, 1]$ such that $|f(x) - f(y)| > C|x - y|$ and hence f is not a Lipschitz function on $[0, 1]$.

Proposition 4.10 *Let A be a non-empty closed subset of \mathbb{R} . If $f : A \rightarrow A$ is a contraction, then there is a fixed point of f , that is, there is a point $a \in A$ such that $f(a) = a$.*

Proof: Since f is a contraction on A , there is $0 < C < 1$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. Fix $x_1 \in A$. Since $f(A) \subseteq A$, we can inductively define a sequence (x_n) in A by $x_{n+1} = f(x_n)$ for $n = 1, 2, \dots$. Notice that we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq C|x_n - x_{n-1}|$$

for all $n = 2, 3, \dots$. This gives

$$|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$$

for $n = 2, 3, \dots$. So, for any $n, p = 1, 2, \dots$, we see that

$$|x_{n+p} - x_n| \leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq |x_2 - x_1| \sum_{i=n}^{n+p-1} C^{i-1}.$$

Since $0 < C < 1$, for any $\varepsilon > 0$, there is N such that $\sum_{i=n}^{n+p-1} C^{i-1} < \varepsilon$ for all $n \geq N$ and $p = 1, 2, \dots$. Therefore, (x_n) is a Cauchy sequence and thus the limit $a := \lim_n x_n$ exists. Since A is closed, we have $a \in A$ and hence f is continuous at a . On the other hand, since $x_{n+1} = f(x_n)$. Therefore, we have $a = f(a)$ by taking $n \rightarrow \infty$. The proof is finished. \square

Remark 4.11 The Proposition 4.10 does not hold if f is not a contraction. For example, if we consider $f(x) = x - 1$ for $x \in \mathbb{R}$, then it is clear that $|f(x) - f(y)| = |x - y|$ and f has no fixed point in \mathbb{R} .

5 Continuous functions defined on intervals

Theorem 5.1 (Intermediate Value Theorem): Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Suppose that $f(a) < z < f(b)$. Then there is c between a and b such that $f(c) = z$.

Proof: Notice that if we consider the function $x \in [a, b] \mapsto f(x) - z$, then we may assume that $z = 0$.

Method I: Let

$$S := \{x \in [a, b] : f(x) \leq 0\}.$$

Notice that the set S is non-empty since $a \in S$ and is bounded. Then by the axiom of completeness, the supremum $c := \sup\{x \in S\}$ exists. Then $c \in [a, b]$ and there is a sequence in S such that $x_n \rightarrow c$. This, together with the continuity of f , imply that $f(c) = \lim_n f(x_n) \leq 0$ since $x_n \in S$. On the other hand, since $b \notin S$, we see that $c \in [a, b)$. Therefore, we can find a sequence (y_n) with $c < y_n < b$ for all n such that $y_n \rightarrow c+$ respectively. By using the continuity of f again, we see that $f(c) = \lim_n f(y_n) \geq 0$ because $y_n \notin S$. Therefore, $f(c) = 0$. The proof is finished.

Method II: Put $x_1 = a$ and $y_1 = b$. Now if $f(\frac{a+b}{2}) = 0$, then the result is obtained. If $f(\frac{a+b}{2}) > 0$, then we set $x_2 = a$ and $y_2 = \frac{a+b}{2}$. Similarly, if $f(\frac{a+b}{2}) < 0$, then we set $x_2 = \frac{a+b}{2}$ and $y_2 = b$. To repeat the same procedure, if there are x_N and y_N such that $f(\frac{x_N+y_N}{2}) = 0$, then the result is shown. Otherwise, we can find a decreasing sequence of closed and bounded intervals $[a, b] = [x_1, y_1] \supseteq [x_2, y_2] \supseteq \dots$ with $\lim(y_n - x_n) = 0$ and $f(x_n) < 0 < f(y_n)$ for all n . Then by the Nested Intervals Theorem, we have $\bigcap_n [x_n, y_n] = \{c\}$ for some $c \in [x_1, y_1] = [a, b]$. Moreover, we have $\lim_n x_n = \lim_n y_n = c$. Then by the continuity of f , we see that $f(c) = \lim f(x_n) = \lim f(y_n)$. Since $f(x_n) < 0 < f(y_n)$ for all n , we have $f(c) = 0$. The proof is finished. \square

Remark 5.2 The assumption of the intervals in the Intermediate Value Theorem is essential. For example, consider $I = [0, 1) \cup (2, 3]$ and define $f : I \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ x - 1 & \text{if } x \in (2, 3]. \end{cases}$$

Then $f(0) < 1 < f(3)$ but $1 \notin f(I)$.

Recall that a non-empty subset I of \mathbb{R} is called an interval if it has one of the following forms.

- (i) \mathbb{R} .
- (ii) $(-\infty, a]$ or $[a, \infty)$ or $(-\infty, a)$ or (a, ∞) for some $a \in \mathbb{R}$.
- (iii) (a, b) or $(a, b]$ or $[a, b)$ or $[a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$.

Lemma 5.3 Let I be a non-empty subset of \mathbb{R} . Suppose that there are different elements in I . Then I is an interval if and only if for any $a, b \in I$ with $a < b$, we have $[a, b] \subseteq I$.

Corollary 5.4 Let $f; [a, b] \rightarrow \mathbb{R}$. Suppose that $M := \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$. Then $f([a, b]) = [m, M]$.

Proof: Notice that if $m = M$, then f is a constant function and hence, the result is clearly true.

Now suppose that $m < M$. It is clear that $f([a, b]) \subseteq [m, M]$ because $m \leq f(x) \leq M$ for all $x \in [a, b]$. For the converse inclusion, notice that since $[a, b]$ is compact, there are x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$. We may assume that $x_1 < x_2$. To apply the Intermediate Value Theorem for the restriction of f on $[x_1, x_2]$, we have $[m, M] \subseteq f([x_1, x_2]) \subseteq f([a, b])$. The proof is finished. \square

Corollary 5.5 Let I be an interval and let $f : I \rightarrow \mathbb{R}$ be a continuous non-constant function. Then $f(I)$ is an interval.

Proof: Notice that by Lemma 5.3, it needs to show that for any $c, d \in f(I)$ with $c < d$ implies that $[c, d] \subseteq f(I)$. Suppose that $a, b \in I$ with $a < b$ satisfy $f(a) = c$ and $f(b) = d$. Notice that $[a, b] \subseteq I$ because I is an interval. If we put $M = \sup_{x \in [a, b]} f(x)$ and $m = \inf_{x \in [a, b]} f(x)$, then by Corollary 5.4, we have

$$[c, d] \subseteq [m, M] = f([a, b]) \subseteq f(I).$$

The proof is finished. \square

Example 5.6 It is impossible to find a continuous surjection from (a, b) onto $(c, d) \cup (e, f)$ where $d \leq e$.

References

- [1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (4th ed), Wiley, (2011).