# MATH 2050A: Mathematical Analysis I (2017 1st term)

Chi-Wai Leung

## 1 Compact Sets in $\mathbb{R}$

Throughout this section, let $(x_n)$ be a sequence in $\mathbb{R}$. Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)$ means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence $(n_k)$ can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, ..\} \mapsto n_k \in \{1, 2, ...\}$.
In this case, note that for each positive integer $N$, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Let us first recall the following two important theorems in real line.

**Theorem 1.1 Nested Intervals Theorem** *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.*

*(i) :* $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

*(ii) :* $\lim_n (b_n - a_n) = 0$.

*Then there is a unique real number $\xi$ such that $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.*

*Proof:* See [1, Theorem 2.5.2, Theorem 2.5.3]. □

**Theorem 1.2 (Bolzano-Weierstrass Theorem)** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

*Proof:* See [1, Theorem 3.4.8]. □

**Definition 1.3** A subset $A$ of $\mathbb{R}$ is said to be *compact* (more precise, *sequentially compact*) if every sequence in $A$ has a convergent subsequence with the limit in $A$.

We are now going to characterize the compact subsets of $\mathbb{R}$. The following is an important notation in mathematics.

**Definition 1.4** A subset $A$ is said to be *closed* in $\mathbb{R}$ if it satisfies the condition:

*if $(x_n)$ is a sequence in $A$ and $\lim x_n$ exists, then $\lim x_n \in A$.*

**Example 1.5** (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}$; $\mathbb{N}$; the empty set $\emptyset$ and $\mathbb{R}$ all are closed subsets of $\mathbb{R}$.

(ii) $(a, b)$ and $\mathbb{Q}$ are not closed.

The following Proposition is one of the basic properties of a closed subset which can be directly shown by the definition. So, the proof is omitted here.

**Proposition 1.6** *Let $A$ be a subset of $\mathbb{R}$. The following statements are equivalent.*

*(i) $A$ is closed.*

*(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.*

The following is an important characterization of a compact set in $\mathbb{R}$. **Warning:** this result is not true for the so-called *metric spaces* in general.

**Theorem 1.7** *Let $A$ be a closed subset of $\mathbb{R}$. Then the following statements are equivalent.*

*(i) $A$ is compact.*

*(ii) $A$ is closed and bounded.*

*Proof:* It is clear that the result follows if $A = \emptyset$. So, we assume that $A$ is non-empty.
For showing $(i) \Rightarrow (ii)$, assume that $A$ is compact.
We first claim that $A$ is closed. Let $(x_n)$ be a sequence in $A$. Then by the compactness of $A$, there is a convergent subsequence $(x_{n_k})$ of $(x_n)$ with $\lim_k x_{n_k} \in A$. So, if $(x_n)$ is convergent, then $\lim_n x_n = \lim_k x_{n_k} \in A$. Therefore, $A$ is closed.
Next, we are going to show the boundedness of $A$. Suppose that $A$ is not bounded. Fix an element $x_1 \in A$. Since $A$ is not bounded, we can find an element $x_2 \in A$ such that $|x_2 - x_1| > 1$. Similarly, there is an element $x_3 \in A$ such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can obtain a sequence $(x_n)$ in $A$ such that $|x_n - x_m| > 1$ for $m \neq n$. From this, we see that the sequence $(x_n)$ does not have a convergent subsequence. In fact, if $(x_n)$ has a convergent subsequence $(x_{n_k})$. Put $L := \lim_k x_{n_k}$. Then we can find a pair of sufficient large positive integers $p$ and $q$ with $p \neq q$ such that $|x_{n_p} - L| < 1/2$ and $|x_{n_q} - L| < 1/2$. This implies that $|x_{n_p} - x_{n_q}| < 1$. It leads to a contradiction because $|x_{n_p} - x_{n_q}| > 1$ by the choice of the sequence $(x_n)$. Thus, $A$ is bounded.
It remains to show $(ii) \Rightarrow (i)$. Suppose that $A$ is closed and bounded.
Let $(x_n)$ be a sequence in $A$. Thus, $(x_n)$. Then the Bolzano-Weierstrass Theorem assures that there is a convergent subsequence $(x_{n_k})$. Then by the closeness of $A$, $\lim_k x_{n_k} \in A$. Thus $A$ is compact.
The proof is finished.
□

## 2 Appendix: Compact sets in $\mathbb{R}$, Part 2

For convenience, we call a collection of open intervals $\{J_\alpha : \alpha \in \Lambda\}$ an *open intervals cover* of a given subset $A$ of $\mathbb{R}$, where $\Lambda$ is an arbitrary non-empty index set, if each $J_\alpha$ is an open

interval (not necessary bounded) and

$$A \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha.$$

**Theorem 2.1 Heine-Borel Theorem:** *For any closed and bounded interval $[a,b]$ satisfies the following condition:*

*(HB) Given any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a,b]$, we can find finitely many $J_{\alpha_1},..,J_{\alpha_N}$ such that $[a,b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$*

*Proof:* Suppose that $[a,b]$ does not satisfy the above Condition $(HB)$. Then there is an open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a,b]$ but it it has no finite sub-cover. Let $I_1 := [a_1,b_1] = [a,b]$ and $m_1$ the mid-point of $[a_1,b_1]$. Then by the assumption, $[a_1,m_1]$ or $[m_1,b_1]$ cannot be covered by finitely many $J_\alpha$'s. We may assume that $[a_1,m_1]$ cannot be covered by finitely many $J_\alpha$'s. Put $I_2 := [a_2,b_2] = [a_1,m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n,b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$;

(b) $\lim_n(b_n - a_n) = 0$;

(c) each $I_n$ cannot be covered by finitely many $J_\alpha$'s.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \le \xi \le b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since $J_{\alpha_0}$ is open, there is $\varepsilon > 0$ such that $(\xi-\varepsilon,\xi+\varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that $a_N$ and $b_N$ in $(\xi-\varepsilon,\xi+\varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N,b_N] \subseteq (\xi-\varepsilon,\xi+\varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property $(c)$ above. The proof is finished.
□

**Remark 2.2** The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.
For example, notice that $\{J_n := (1/n,1) : n = 1,2...\}$ is an open interval covers of $(0,1)$ but you cannot find finitely many $J_n$'s to cover the open interval $(0,1)$.

The following is a very important feature of a compact set.

**Theorem 2.3** *Let $A$ be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

*(i) For any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $A$, we can find finitely many $J_{\alpha_1},..,J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$.*

*(ii) $A$ is compact.*

*(iii) $A$ is closed and bounded.*

*Proof:* The result will be shown by the following path

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).$$

For $(i) \Rightarrow (ii)$, assume that the condition $(i)$ holds but $A$ is not compact. Then there is a sequence $(x_n)$ in $A$ such that $(x_n)$ has no subsequent which has the limit in $A$. Put $X = \{x_n : n = 1, 2, ...\}$. Then $X$ is infinite. Also, for each element $a \in A$, there is $\delta_a > 0$ such that $J_a := (a-\delta_a, a+\delta_a) \cap X$ is finite. Indeed, if there is an element $a \in A$ such that $(a-\delta, a+\delta) \cap A$ is infinite for all $\delta > 0$, then $(x_n)$ has a convergent subsequence with the limit $a$. On the other hand, we have $A \subseteq \bigcup_{a \in A} J_a$. Then by the compactness of $A$, we can find finitely many $a_1, ..., a_N$ such that $A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. So we have $X \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Then by the choice of $J_a$'s, $X$ must be finite. This leads to a contradiction. Therefore, $A$ must be compact.
The implication $(ii) \Rightarrow (iii)$ follows from Theorem 1.7 at once.
It remains to show $(iii) \Rightarrow (i)$. Suppose that $A$ is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha \in \Lambda}$ be an open intervals cover of $A$. Notice that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since $A$ is closed by using Proposition 1.6. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha \cup \bigcup_{x \in [a,b] \setminus A} I_x.$$

Using the Heine-Borel Theorem 2.1, we can find finitely many $J_\alpha$'s and $I_x$'s, say $J_{\alpha_1}, ..., J_{\alpha_N}$ and $I_{x_1}, ..., I_{x_K}$, such that $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of $I_x$. Therefore, we have $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ and hence $A$ is compact.
The proof is finished. □

**Remark 2.4** In fact, the condition in Theorem 2.3(i) is the usual definition of a *compact set* for a general topological space. More precise, if a set $A$ satisfies the Definition 1.4, then $A$ is said to be *sequentially compact.* Theorem 2.3 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of $\mathbb{R}$. However, these two notation are different for a general topological space.

**Strongly recommended:** take the courses: MATH 3060; MATH3070 for the next step.

# References

[1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (4*th ed*), Wiley, (2011).