TA's solution to 2060B homework 3*

p.187 Q3. (2.5 marks)

Since

$$0 \leq \left| x^2 \sin\left(\frac{1}{x}\right) \right| \leq x^2$$

for $x \in (0,1]$, we have $\lim_{x\to 0} f(x) = 0$ by squeeze theorem. Also, by the continuity of g at the point 0, we have $\lim_{x\to 0} g(x) = g(0) = 0$.

On the other hand, $\lim_{x\to 0} f(x)/g(x)$ does not exist, because for $x_n := \frac{1}{2n\pi}$, $y_n := \frac{1}{2n\pi+0.5\pi}$, we have $x_n, y_n \in (0,1]$, $x_n, y_n \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} f(x_n)/g(x_n) = \lim_{n\to\infty} \sin(2n\pi) = 0, \text{ while}$$

$$\lim_{n\to\infty} f(y_n)/g(y_n) = \lim_{n\to\infty} \sin(2n\pi + 0.5\pi) = 1.$$

p.196 Q9. (2.5 marks)

†By differentiating the sine function repeatedly we get

$$\frac{d^{2k}}{dx^{2k}} \sin x = (-1)^k \sin x, \; \frac{d^{2k+1}}{dx^{2k+1}} \sin x = (-1)^k \cos x, k \geq 0.$$

By Taylor's Expansion Theorem,

$$\sin x = \sin x_0 + (\cos x_0)(x-x_0) + \cdots + \frac{f_n(x_0)}{n!}(x-x_0)^n + \frac{f_{n+1}(c_n)}{(n+1)!}(x-x_0)^{n+1},$$

where each f_k is one of $\pm\sin$ or $\pm\cos$, and c_n is some mean value between x and x_0. Using $|\sin c_n|$, $|\cos c_n| \leq 1$, the remainder is bounded by

$$\frac{|x-x_0|^{n+1}}{(n+1)!}$$

which tends to 0 as $n \to \infty$.‡

*It is strange that a number of you have done p. 196 Q4 instead of Q9. I try to give some marks in this situation. Starting from next time let us agree that the course website is the single source of truth w.r.t. homework question no.

†This solution is adapted from the work of MATH2060A, Solution to Assignment 3, p.1. (https://www.math.cuhk.edu.hk/course_builder/1819/math2060a/2060%20Solution%203%202019.pdf)

‡A simple way to get this is by ratio test (textbook 3.2.11 Theorem).

p.196 Q10. (5 marks)

A well-written solution can be found in MATH2060A, Solution to Assignment 3, p.2. We make some observation:

- It seems that unavoidably we have to show $\lim_{x\to 0} \frac{P(1/x)}{e^{1/x^2}} = 0$ for any real-coefficient polynomial $P(z)$. Observe that a change of variable $y = 1/x$ may make the problem easier (but be careful about the $\pm$ signs: $y \to -\infty$ if $x \to 0^-$). By using absolute value (we go to use squeeze theorem at the last step) and triangle inequality, it suffices to deal with

$$\lim_{y\to+\infty} \frac{\sum_{i=0}^{N} a_i y^i}{e^{y^2}},$$

  where $a_i \geq 0$ and $a_N > 0$. By considering another application of squeeze theorem, we can replace e^{y^2} by e^y too, as $1/e^{y^2} \leq 1/e^y$ for $y > 1$. We can now use L'Hospital's rule easily.

- Most people use L'Hospital's rule casually, but when we are in this course, in view of the "if-then" logic flow[§], the following presentation is not without criticism:

$$\begin{aligned}\lim_{y\to+\infty} \frac{\sum_{i=0}^{N} a_i y^i}{e^y} &= \lim_{y\to+\infty} \frac{a_1 + 2a_2 y + \cdots + N a_N y^{N-1}}{e^y} \\ &= \cdots = \lim_{y\to+\infty} \frac{N! a_N}{e^y} = 0.\end{aligned}$$

- Note that the c in the remainder term in Taylor theorem, $\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$, not only depends on x and x_0 but also depends on n (so by careful when we try to let $n \to \infty$).

[§] i.e. statement like: if $\lim \frac{f'}{g'} = L$, then $\lim \frac{f}{g} = L$.