TA's solution to 2060B homework 1[*]

p.179 Q4. (3 marks)

Now f is differentiable everywhere on $\mathbb{R}$ and f' is given by

$$f'(x) = -2\sum_{i=1}^{n}(a_i - x).$$

Since f is differentiable on $\mathbb{R}$, therefore if f has a relative extremum at some $c \in \mathbb{R}$, then $f'(c)$ must be zero[†]. Solving $f'(x) = 0$, we get $x_0 := \frac{1}{n}\sum_{i=1}^{n} a_i$. Hence, x_0 is the only possible point of relative extremum for f.

Since

$$\begin{aligned} f'(x) &= -2\sum_{i=1}^{n}(a_i - x) \\ &= 2nx - 2\sum_{i=1}^{n} a_i \\ &= 2n(x - x_0), \end{aligned}$$

we see that $f'(x) < 0\ \forall x < x_0$, while $f'(x) > 0\ \forall x > x_0$. Therefore[‡], f is strictly decreasing on $(-\infty, x_0)$ and strictly increasing on (x_0, ∞), while being continuous at x_0. This shows that x_0 is the unique point of relative minimum for f.

p.179 Q7. (3 marks)

Define $f : (0, \infty) \to \mathbb{R}$ by $f(t) := \ln t$ and fix $x > 1$. Since f is continuous on $[1, x]$ and differentiable on $(1, x)$, by Mean Value Theorem $\exists\ ? \in (1, x)$ such that

$$f(x) - f(1) = f'(?)(x - 1).$$

[*]If you find any issue about the solution or the marking, please feel free to contact TA (cychan [AT] math.cuhk.edu.hk).

[†]This is textbook 6.2.1 Interior Extremum Theorem. If we remember the very definition of derivative, then doing the proof on site may be easier than recalling the statement of that theorem.

[‡]Similar to the previous footnote.

By the hint we have $f'(?) = \frac{1}{?}$. As $? \in (1, x)$, it follows that

$$\frac{1}{x} < f'(?) < \frac{1}{1} = 1.$$

Noting that $f(1) = 0$ and $x - 1 > 0$, combining the results we get

$$\frac{x-1}{x} < \ln x < x - 1,$$

which was to be demonstrated.

p.179 Q10. (4 marks)

Since

$$\begin{aligned}\lim_{h\to 0} \frac{g(h) - g(0)}{h} &= \lim_{h\to 0} \frac{h + 2h^2 \sin \frac{1}{h}}{h} \\ &= \lim_{h\to 0} (1 + 2h \sin \frac{1}{h}) \\ &= 1,^{\S}\end{aligned}$$

we have $g'(0) = 1$. On the other hand, by basic differentiation rules, we get for $x \neq 0$,

$$g'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}.$$

Note that every neighborhood of 0 (no matter how small it is) contains points of the form $1/2n\pi$ and $1/(2m\pi + \frac{\pi}{2})$, where $n, m \in \mathbb{N}$. Since

$$g'(\frac{1}{2n\pi}) = -1 < 0$$

and

$$g'(\frac{1}{2m\pi + \frac{\pi}{2}}) = 1 + \frac{4}{2m\pi + \frac{\pi}{2}} > 0,$$

the result follows.

§ $\lim_{h\to 0} 2h \sin \frac{1}{h} = 0$ because when $h \to 0$, $\left|\sin \frac{1}{h}\right|$ is bounded by 1, while $2h \to 0$.