MATH 2060 Mathematical Analysis II
HW5 suggested solution
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P.246 Q4:

$$\lim_n x^n = \begin{cases} 0 & \text{when } x \in [0,1)\ , \\ 1 & \text{when } x = 1, \\ +\infty & \text{when } x > 1. \end{cases}$$

Thus,

$$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0 & \text{when } x \in [0,1)\ , \\ \frac{1}{2} & \text{when } x = 1, \\ 1 & \text{when } x > 1. \end{cases}$$

P.247 Q14:

Denote $f_n(x) = \dfrac{x^n}{1+x^n}$ and

$$f(x) = \begin{cases} 0 & \text{when } x \in [0,1)\ , \\ \frac{1}{2} & \text{when } x = 1, \\ 1 & \text{when } x > 1. \end{cases}$$

If $b \in (0,1)$, on $[0,b] \subset [0,1)$. $f_n(x)$ converge to $f(x) = 0$ pointwisely on $[0,b]$. Since $\lim_n b^n = 0$, for any $\epsilon > 0, \exists N \in \mathbb{N}$ such that $0 < b^n < \epsilon\ , \forall n > N$. Thus, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $x \in [0,b]$, $n > N$,

$$|f_n(x) - f(x)| = \frac{x^n}{1+x^n} < b^n < \epsilon.$$

So, the convergence is uniform on $[0,b]$. But the convergence is non-uniform on $[0,1]$. We can take $n_k = k$, $x_k = (1 - \frac{1}{k})$.

$$|f_k(x_k) - f(x_k)| = \frac{(1-1/k)^k}{1+(1-1/k)^k} \to \frac{e^{-1}}{1+e^{-1}} > 0, \text{ as } k \to \infty.$$

Or using the theorem in the book, assume the convergence is uniform on $[0,1]$, since $\{f_n\}$ are all continuous function on $[0,1]$, the limit function f is also continuous. Contradiction arised.

P.247 Q22:

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $1/n < \epsilon$. Thus,

$$|f_n(x) - f(x)| = \frac{1}{n} < \epsilon \text{ for all } x \in \mathbb{R},\ \forall n > N.$$

f_n^2 converges to f^2 poinwisely. So it suffices to show that f_n^2 does not converge uniformly to f^2 on $\mathbb{R}$. We take $n_k = k$, $x_k = k$. So,

$$|f_k^2(x_k) - f^2(x_k)| = |\frac{2k}{k} + \frac{1}{k^2}| > 1.$$

Thus, f_n^2 does not convrge uniformly on $\mathbb{R}$.

P.247 Q23:

Since $\{f_n\}, \{g_n\}$ are uniformly bounded, there exists $M > 0$ such that

$$|f_n(x)|, |g_n(x)| \leq M \ , \forall x \in A.$$

$\{f_n\}, \{g_n\}$ converge uniformly to f and g respectively. So for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in A, n > N$

$$|f_n(x) - f(x)| < \epsilon \quad \text{and} \quad |g_n(x) - g(x)| < \epsilon.$$

Also,

$$|f(x)|, |g(x)| \leq M, \ \forall x \in A.$$

Thus, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in A, n > N$

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &< 2M\epsilon. \end{aligned}$$