MATH 2060 Mathematical Analysis II
Tutorial Class 8
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1. (a) Define pointwise and uniform convergence of a sequence of functions.

   (b) Let $A \subset \mathbb{R}$ and $f_n, f : A \to \mathbb{R}$. Show that f_n does not converge uniformly to f on A if and only if there exists $\epsilon_0 > 0$, a subsequence $\{f_{n_k}\}$ and a sequence $\{x_k\}$ in A such that $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0$ for all $k \in \mathbb{N}$.

   (c) Show that the convergence of $f_n(x) = x + \dfrac{x^2}{n}$ is not uniform on $\mathbb{R}$.

   (d) Show that the convergence of $f_n(x) = x + \dfrac{nx}{1 + nx^2}$ is not uniform on $[0, \infty)$.

2. (a) Let $f_n, f : A \to \mathbb{R}$. Show that $\{f_n\}$ converge uniformly to f on A if and only if

$$\sup\{|f_n(x) - f(x)| : x \in A\} \to 0 \text{ as } n \to 0.$$

   (b) If each f_n is continuous on A, show that f is also continuous on A.

   (c) Show that $f_n(x) = \dfrac{x}{1 + nx^2}$ converge uniformly on $\mathbb{R}$.

3. Let $f_n, f : A \to \mathbb{R}$. Suppose each f_n is uniformly continuous on A and $\{f_n\}$ converge uniformly to f on A.

   (a) Show that f is uniformly continuous on A.

   (b) Prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$, if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$

4. Let $f_n, f : [a, b] \to \mathbb{R}$ such that $\{f_n\}$ converge to f pointwisely on $[a, b]$. Suppose each f_n is differentiable, f is continuous and there exist $M > 0$ such that $|f_n'| < M$ on $[a, b]$ for all n, prove that $\{f_n\}$ converge to f uniformly. (Pastpaper 2004-2005)