Suggested solution of HW4

P.215 Q17:

f is continuous function on $[a,b]$. Since $f(x)>0$ on $[a,b]$, by max-min theorem, there exists $m=\inf\{f(x):x\in[a,b]\}$ and $M=\sup\{f(x):x\in[a,b]\}$ such that

$$m\le f(x)\le M \quad \forall x\in[a,b].$$

So, we have

$$m\int_a^b g\le\int_a^b fg\le M\int_a^b g.$$

Now, we check that $\int_a^b g>0$. By max-min theorem, there exists $c\in[a,b]$ such that $g(c)=m'=\inf\{g(x):x\in[a,b]\}>0$, $g(x)\ge m'$ for all $x\in[a,b]$. By ordering property of Riemann integral, we have

$$\int_a^b g\ge\int_a^b m'=(b-a)m'>0.$$

Thus, we have

$$m\le\frac{\int_a^b fg}{\int_a^b g}\le M.$$

Apply intermediate value theorem to f, there exists $c'\in[a,b]$ such that $\frac{\int_a^b fg}{\int_a^b g}=f(c')$.

Noted that $g(x)>0$ is essential. If we choose $g(x)=1-x,\ x\in[0,2]$, $f(x)=1+x, x\in[0,2]$, it is clear that

$$\int_0^2 fg>0 \quad\text{and}\quad f(c)\int_0^2 g=0,\forall\, c\in[0,2].$$

P.225 Q21:

(a) Let $t\in\mathbb{R}$ be given, $(tf\pm g)^2\ge 0$ for all $x\in[a,b]$. By ordering properties of Riemann integral, we have

$$\int_a^b (tf\pm g)^2\ge\int_a^b 0=0.$$

(b) By part (a), we have

$$\int_a^b (t^2f^2\pm 2tfg+g^2)\ge 0 \quad ,\forall t\in\mathbb{R}.$$

Since $f,g\in R[a,b]$, it follows that $f^2,g^2,fg\in R[a,b]$. Thus,

$$t\int_a^b f^2+\frac{1}{t}\int_a^b g^2\ge 2\left|\int_a^b fg\right| \quad ,\forall t>0.$$

(c) If $\int_a^b f^2 = 0$, part (b) implies that

$$\frac{1}{t}\int_a^b g^2 \geq 2\left|\int_a^b fg\right| \quad ,\forall t > 0.$$

Letting $t \to +\infty$, we get $\int_a^b fg = 0$.

(d) By considering the function $|f|, |g|$, we can assume f, g are non-negative function on $[a, b]$.

If $\int_a^b f^2 = 0$, $\int_a^b fg = 0$ by the result of part(c). So,

$$(\int_a^b fg)^2 \leq (\int_a^b f^2)\,(\int_a^b g^2) \quad \text{holds.}$$

If $\int_a^b f^2 > 0$, since we have

$$t^2\int_a^b f^2 + 2t\int_a^b fg + \int_a^b g^2 \geq 0 \quad ,\forall t \in \mathbb{R}.$$

By testing the discriminant of a quadratic equation, we obtain

$$D = \left(2\int_a^b fg\right)^2 - 4\left(\int_a^b f^2\right)\left(\int_a^b g^2\right) < 0.$$

Result follows.