THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Tutorial Note 7
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Fundamental theorems (continued)

Proposition. If (x_n) in a Banach space is such that $(f(x_n))$ is bounded for all $f \in X^*$, than $(\|x_n\|)$ is bounded.

Example 6.8. Let X be an infinite-dimensional Banach space and $\mathcal{B}$ a normalized Hamel basis in X. We choose a sequence (e_n) in $\mathcal{B}$ and define $f(e_n) = n$ for $n \in \mathbb{N}$ and $f(e) = 0$ for $e \in \mathcal{B} \setminus \{e_n : n \in \mathbb{N}\}$. By extending f linearly to the entire space X, we obtain an unbounded real linear functional, which we denote by the same symbol f.

By the closed graph theorem, the graph $G(f)$ is not closed in $X \times \mathbb{R}$ in the product topology defined by the norm

$$\|(x,t)\| = \|x\| + |t|, \quad x \in X, t \in \mathbb{R}.$$

$G(f)$ is not a Banach space since a subspace of a Banach space is complete if and only if it's closed. We define $T : G(f) \to X$ by $T(x, f(x)) = x$ for $x \in X$. Clearly, T is a bijection. The operator T is bounded, because

$$\frac{\|x\|}{\|(x, f(x))\|} \leq 1, \quad \forall x \in X, x \neq 0.$$

Inasmuch as

$$\frac{\|(x, f(x))\|}{\|x\|} \geq \frac{|f(x)|}{\|x\|}, \quad \forall x \in X, x \neq 0,$$

and f is unbounded, the operator T^{-1} is unbounded.

An example to illustrate UBT. Let $T_n = S^n$, where the operator $S : l^2 \to l^2$ is defined by

$$(x_1, x_2, x_3, \cdots) \mapsto (x_2, x_3, x_4, \cdots).$$

Find a bound for $\|T_n x\|$; find $\lim_{n\to\infty} \|T_n x\|, \|T_n\|$ and $\lim_{n\to\infty} \|T_n\|$.

Answers: $\|T_n x\| \leq \|x\|; 0; 1; 1.$

Example. The completeness of X is essential and cannot be dropped in Uniform Boundedness Theorem. Let's consider the subspace $X \subset l^\infty$ consisting of all $x = (x_n)$ such that $x_n = 0$ for $n \geq N \in \mathbb{N}$, where N depends on x, and let T_n be defined by

$$T_n x = n x_n.$$

It can be checked that X is incomplete. And for any $x \in X$,

$$|T_n x| = \begin{cases} n|x_n| \leq N\|x\|, & n < N, \\ 0, & n \geq N. \end{cases}$$

So that $|T_n x|$ is bounded for every $x \in X$. But $\{\|T_n\|\}$ is unbounded since

$$\|T_n\| = n.$$