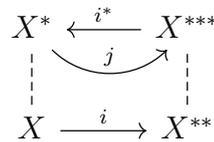


Recall

Projection and decomposition in Banach spaces

Let X be a Banach space. A bounded linear operator $P \in B(X, X)$ is a *projection* if $P^2 = P$ (*idempotent*). For any projection P , we have $X = \text{Im}(P) \oplus \text{Ker}(P)$. A closed subspace $M \subset X$ is *complemented* if there exists a closed subspace $N \subset X$ such that $X = M \oplus N$.

- closed subspace $M \subset X$ complemented $\iff \exists$ projection P with $\text{Im}(P) = M$.
- Any finite dimensional subspace in normed space is complemented.
- c_0 is not complemented in ℓ^∞ and $c_0 \neq X^*$ for any normed space X .
- (Dixmier) Let $i: X \rightarrow X^{**}$ and $j: X^* \rightarrow X^{***}$ be the natural embeddings.



By viewing X as a subspace of X^{**} , the projection $D := j \circ i^*$ implies

$$X^{***} = \text{Im}(D) \oplus \text{Ker}(D) = X^* \oplus X^\perp,$$

where the annihilator $X^\perp := \{y \in X^{***} : y(x) = 0, \forall x \in X\}$. In particular, letting $X = c_0$, we have $(\ell^\infty)^* = \ell^1 \oplus c_0^\perp$.

- By further considering norms on direct sum, we denote $X = M \oplus_{\ell_1} N$ if $X = M \oplus N$ and $\|x\| = \|y\| + \|z\|$ for every $x = y + z$ with $y \in M, z \in N$. Then

$$(\ell^\infty)^* = \ell^1 \oplus_{\ell_1} c_0^\perp.$$

About norm and inner product

Example 1. $C[0, 1]$ with sup-norm $\|\cdot\|_\infty$ is not an inner product space.

Proof. We prove by showing $\|\cdot\|_\infty$ does not satisfy Parallelogram Law.

Consider functions $x(t) = 1$ and $y(t) = t$ for $t \in [0, 1]$. Then

$$\|x\|_\infty = 1 \quad \text{and} \quad \|y\|_\infty = 1$$

while

$$\|x + y\|_\infty = \sup_{t \in [0,1]} (1 + t) = 2 \quad \text{and} \quad \|x - y\|_\infty = \sup_{t \in [0,1]} (1 - t) = 1.$$

Hence

$$\|x + y\|_\infty^2 + \|x - y\|_\infty^2 = 5 \neq 4 = 2(\|x\|_\infty^2 + \|y\|_\infty^2).$$

□

Theorem 2 (Polarization identities). *If X is a real inner product space, then*

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2), \quad \forall x, y \in X. \quad (1)$$

For a complex inner product space X , we have

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2), \quad \forall x, y \in X. \quad (2)$$

Proof. (i) (Real case) By the bilinearity of real inner product,

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \quad (3)$$

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2. \quad (4)$$

Then (1) is obtained by simplifying (3) - (4).

(ii) (Complex case) By the sesquilinearity of complex inner product,

$$\|x + y\|^2 = \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 = \|x\|^2 + 2\Re\langle x, y \rangle + \|y\|^2 \quad (5)$$

$$\|x - y\|^2 = \|x\|^2 - 2\Re\langle x, y \rangle + \|y\|^2 \quad (6)$$

$$\|x + iy\|^2 = \|x\|^2 - i\langle x, y \rangle + i\overline{\langle x, y \rangle} + \|y\|^2 = \|x\|^2 + 2\Im\langle x, y \rangle + \|y\|^2 \quad (7)$$

$$\|x - iy\|^2 = \|x\|^2 - 2\Im\langle x, y \rangle + \|y\|^2. \quad (8)$$

Then $\Re\langle x, y \rangle$ follows from (5) - (6) and $\Im\langle x, y \rangle$ follows from (7) - (8), thus (2). □

Example 3. Let X be a normed space with norm $\|\cdot\|$. Then

$\|\cdot\|$ is induced by an inner product $\iff \|\cdot\|$ satisfies the Parallelogram Law.

Proof. \implies is the property of inner product.

Since it is fun and meaningful to check \impliedby by ourselves, we omit the details but leave a possible sketch: Define $\langle \cdot, \cdot \rangle: X \times X \rightarrow \mathbb{C}$ as (2). Then $\langle x, x \rangle \geq 0$ and $\langle y, x \rangle = \overline{\langle x, y \rangle}$ directly follows from (2). As for the linearity on the first argument, firstly we can prove the additivity $\langle x + \tilde{x}, y \rangle = \langle x, y \rangle + \langle \tilde{x}, y \rangle$ via Parallelogram Law. To achieve $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, we may check in the following order

$$\text{additivity} \rightarrow \alpha \in \mathbb{N} \xrightarrow{\text{"-"}} \alpha \in \mathbb{Z} \xrightarrow{\text{"n/m"}} \alpha \in \mathbb{Q} \xrightarrow{\text{continuity}} \alpha \in \mathbb{R} \xrightarrow{\text{"i"}} \alpha \in \mathbb{C}.$$

□