Recall

Fundamental theorems

Open Mapping Theorem Let X, Y be **Banach** spaces. Let $T \in B(X, Y)$ be **surjective**. Then T is a open mapping.

- Let $T \in B(X, Y)$ be a linear isomorphism. Then T^{-1} is bounded.
- For $A \in B(X, Y)$, denote the image of A by $\operatorname{Im}(A)$.
  Let $T, K \in B(X, Y)$. $\{ \operatorname{Im}(T)$ closed & $\dim \operatorname{Im}(K) < \infty \} \implies \operatorname{Im}(T + K)$ is closed. However, if weaken $\dim \operatorname{Im}(K) < \infty$ to $\operatorname{Im}(K)$ being closed, then $\implies$ may not hold.

Closed Graph Theorem Let X, Y be **Banach** spaces. Let $T\colon X \to Y$ be a linear operator. Then T is bounded $\iff$ T has closed graph, i.e., $\mathscr{G}(T) := \{(x, Tx)\colon x \in X\}$ is closed in $(X \oplus Y, \|\cdot\|_\infty)$.

- Let $G\colon Y^* \to X^*$ be a w^*–w^* continuous linear map. Then
  (i) $G \in B(Y^*, X^*)$. (ii) $\exists\, T \in B(X, Y)$ such that $T^* = G$.

Uniform Boundedness Theorem Let X be a **Banach** space and Y be a normed space. Let $(T_i)_{i\in I}$ be a family of bounded linear operators from X to Y. Suppose for all $x \in X$, we have $\sup_{i\in I}\|T_i(x)\| < \infty$. Then $\sup_{i\in I}\|T_i\| < \infty$.

- Let $(T_n)_{n=1}^\infty \in B(X, Y)$. Suppose $\lim_{n\to\infty} T_n(x)$ exists in Y for all $x \in X$. Then there exists $T \in B(X, Y)$ such that $T(x) = \lim_{n\to\infty} T_n(x)$ for all $x \in X$ and $\|T\| \le \liminf_{n\to\infty}\|T_n\|$.
- Weakly convergent sequences in normed spaces are bounded.

To prove the above theorems, it is essential to exploit the completeness of Banach spaces via *Baire Category Theorem*.

Examples to illustrate the theorems

To prove a map is open, Open Mapping Theorem is sufficient but not necessary. The following example shows that a surjective bounded mapping $T\colon X \to Y$ may be open while X is not a Banach space.

Recall that a topological vector space is a vector space equipped with a (Tychonoff) topology in which the vector space operations are continuous.

Example 1. Let X be a nontrival topological vector space. Let f be any nonzero continuous functional on X, i.e, $0 \ne f \in X^*$. Then f is open.

Proof. By the translation-invariance of the vector topology on X, it suffices to prove $f(V)$ is open in scalar field $\mathbb{C}$ for every open neighborhood of $0 \in X$.

Since $f \ne 0$, there exists $x_0 \in X$ such that $f(x_0) \ne 0$. By normalizing x_0 (w.r.t. f) to $x^* = x_0/f(x_0)$, we have $f(x^*) = 1$. Take any $y \in f(V)$. Then there exists $x \in V$ with $f(x) = y$.

Since both of the scalar multiplication and the addition are continuous and V is open, the map $r \mapsto x + rx^*$ is continuous and there exists $\delta > 0$ such that $x + rx^* \in V$ for all $|r| < \delta$. Then $y + r = f(x + rx^*) \in f(V)$ for all $|r| < \delta$. Then y is an interior point of $f(V)$, thus $f(V)$ is open for y is arbitrary. □

The Banach space assumptions both on the domain X and the range Y in Open Mapping Theorem and Closed Graph Theorem are essential.

Example 2. Let $X = (\ell^1, \|\cdot\|_1)$ and $Y = (\ell^1, \|\cdot\|_\infty)$. Let T be the identity map from X onto Y. Then T is continuous but not open.

Proof. Let $x \in \ell^1$. Since $\|x\|_\infty \leq \|x\|_1$, we have T is bounded.

For $n \in \mathbb{N}$, define $x_n(i) = \begin{cases} 1 & i \leq n \\ 0 & i > n \end{cases}$. Let $x_n = (x_n(i))_{i=1}^\infty$. Since $\|x_n\|_\infty = 1$ for all $n \in \mathbb{N}$ while $\|x_n\|_1 = n \to \infty$ as $n \to \infty$, we have T^{-1} is not bounded.

Next we check that Y is not complete. For $n \in \mathbb{N}$, define $x_n(i) = \begin{cases} 1/i & i \leq n \\ 0 & i > n \end{cases}$. Let $x_n = (x_n(i))_{i=1}^\infty$. Then $x_n \in \ell_1$ for all $n \in \mathbb{N}$. Moreover, $(x_n)_{n=1}^\infty$ is a Cauchy sequence with respect to $\|\cdot\|_\infty$. Suppose on the contrary that there exists $x = (x(i))_{i=1}^\infty \in \ell_1$ such that $x_n \to x$ in $\|\cdot\|_\infty$. Since $|x(i) - x_n(i)| \leq \|x - x_n\|_\infty$ for all $i \in \mathbb{N}$, we have $x(i) = 1/i$ for all $i \in \mathbb{N}$. Then $\sum_{i=1}^\infty x(i) = \sum_{i=1}^\infty 1/i = \infty$ contradicts $x \in \ell_1$. Hence $(\ell_1, \|\cdot\|_\infty)$ is not complete. □