1 (2021 Home Test 1 Q1)**.** Let $f(x) = \text{sgn}(\sin \frac{\pi}{x})$ for $x \neq 0$ and $f(0) = 0$, where sgn denotes the sign function. Show that f is Riemann integrable over $[-1, 1]$ and find $\int_{-1}^{1} f(x)dx$.

2 (2021 Home Test 1 Q2)**.** Let f be a continuous real-valued function defined on $\mathbb{R}$.

(a) Suppose that there are constants c_0 and c_1 such that

$$\lim_{x\to 0} \frac{f(x) - c_0 - c_1 x}{x} = 0.$$

Show that $f'(0)$ exists.

(b) Suppose that f is a C^1-function and there are constants c_0, c_1 and c_2 such that

$$\lim_{x\to 0} \frac{f(x) - c_0 - c_1 x - c_2 x^2}{x^2} = 0.$$

Does it imply that the second derivative of f at 0 exist? Prove your assertion.

3 (2021 Home Test 1 Q3)**.** Let $f : (0, 1) \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} \frac{1}{p} & \text{if } \; x = \frac{q}{p} \text{ and } p, q \text{ are relatively prime positive integers;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

a) Describe the continuity of f.

b) Describe the differentiability of f.

Justify your answer by using the definitions.

4 (Motivated from 1920 Home Test 1 Q2)**.** Recall that a function $s : [0,1] \to \mathbb{R}$ is a step function over $[0,1]$ if there exists a partition $P := \{x_i\}_{i=0}^k \subset [0,1]$ such that s is constant over (x_{i-1}, x_i).

(a) Let $f \in \mathcal{R}([0,1])$. Show that there exists a sequence of step functions (s_n) over $[0,1]$ such that $s_n \leq s_{n+1}$ pointwise for all $n \in \mathbb{N}$ and $\lim_n \int_0^1 s_n = \int_0^1 f$.

(b) Let $f \in \mathcal{C}([0,1])$, that is f is continuous. Show that there exists a sequence of step functions (s_n) uniformly approximating f, that is, $\lim_n \sup_{x\in[0,1]} |s_n(x) - f(x)| = 0$. Hence, show that the sequence also satisfies $\lim_n \int_0^1 s_n = \int_0^1 f$.

(c) Suppose $f \in \mathcal{R}([0,1])$. Is it always true that f is uniformly approximated by step functions, that is, can the assumption in (b) be relaxed to only integrable functions?