Unless otherwise specified, we always use I to denote an open interval; logarithmic expressions (with log) are in the natural base. If we write (a, b) or $[a, b]$, it is always the case that $a < b \in \mathbb{R}$.

1 L'Hospital's Rule

Theorem 1.1 (Generalized Cauchy Mean Value Theorem). *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b). Suppose $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}$$

Theorem 1.2 (L'Hospital Rule). *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b) where $-\infty \leq a < b \leq \infty$ (turn closed brackets to open brackets for $\pm\infty$). Suppose $g'(x) \neq 0$ for all $x \in (a, b)$. Further suppose $L := \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists. Then we have*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

if either of the following is satisfied:

a) $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$

b) $\lim_{x\to a^+} g(x) = \infty$ *or* $\lim_{x\to a^+} g(x) = -\infty$

Similar statements can be made for left-handed limits and two-sided limits.

Quick Practice

1. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose $f'(x) \neq 0$ for all $x \in (a, b)$. Show that
   (a) f is injective.
   (b) Suppose $f'(c) > 0$ for some $c \in (a, b)$. Then f is strictly increasing.

2. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable such that $g'(x) \neq 0$ for all $x \in (a, b)$. Fix $c \in (a, b)$. Suppose $\lim_{x\to c^+} f(x) = \lim_{x\to c^+} g(x) = 0$. Show that the L'Hospital Rule holds, that is, if $L := \lim_{x\to c^+} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to c^+} \frac{f(x)}{g(x)} = L$ *using* the Cauchy Mean Value Theorem.

3. Let $f(x) := \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ and let $g(x) := \sin x$ for all $x \in \mathbb{R}$.
   (a) Show that $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$ but $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.
   (b) Some says that the above example violates the L'Hospital Rule. Do you agree? Explain your answer.

4. Evaluate the following limits:
   a) $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$
   b) $\lim_{x\to 0^+} x^3 \log x$
   c) $\lim_{x\to\infty} x^3 e^{-x}$
   d) $\lim_{x\to 0^+} \sqrt{x} \log(x)$
   e) $\lim_{x\to\infty} x^{1/x}$
   f) $\lim_{x\to\infty} (1 + 3/x)^x$

5. (Very Tricky Question). Let f be differentiable on $(0, \infty)$. Suppose $L := \lim_{x\to\infty}(f(x) + f'(x))$ exists. Show that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$

6. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable such that $g'(x) \neq 0$ for all $x \in (a, b)$. Fix $c \in (a, b)$. Prove the L'Hospital Rule under the assumption that $\lim_{x\to c^+} g(x) = \infty$.

2 More on Convex Functions

Definition 2.1. Let $f : I \to \mathbb{R}$ where I is an interval. Then f is a convex function if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$

Theorem 2.2. *Suppose $f : I \to \mathbb{R}$ is twice-differentiable on an open intervals. Then f is convex on I if and only if $f'' \ge 0$ on I.*

Quick Practice

1. Let $p > 0$. Define $f(x) := x^p$ on $(0, \infty)$. Find all values of p such that f is convex on $(0, \infty)$

2. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function.
   (a) Show that f is convex on $[0, \infty)$ if it is convex on $(0, \infty)$
   (b) Suppose f is convex, increasing on $[0, 1]$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := f(|x|)$. Show that g is a convex function.
   (c) Let $p \ge 1$. Show that $g(x) := |x|^p$ is convex on $\mathbb{R}$ but is not differentiable on $\mathbb{R}$ in general.
   (d) Let $p \in (0, 1)$. Show that $g(x) := |x|^p$ is a concave function on $[0, \infty)$, that is, $g(tx + (1-t)y) \ge tg(x) + (1-t)g(y)$ for all $x, y \in [0, \infty)$ and $t \in [0, 1]$. Is it true that g is concave on $\mathbb{R}$?

3. In this exercise, we would be showing the Hölder's inequality: that is, let $p, q \ge 1$ be having the property that $p^{-1} + q^{-1} = 1$ (we say that p, q are conjugate exponents of each other). Then for all finite list of real numbers $(x_i)_{i=1}^k$ and $(y_i)_{i=1}^k$, we have

$$\sum_{i=1}^{k} |x_i y_i| \le (\sum_{i=1}^{k} |y_i|^p)^{1/p} (\sum_{i=1}^{k} |x_i|^q)^{1/q}$$

   Observe that this is a generalization to the Cauchy-Schwarz inequality where $p = q = 2$.
   (a) Show that $x \mapsto \exp(x)$ is a convex function on $\mathbb{R}$.
   (b) Using the above convexity, show that for all $r, s \ge 0$ and $p, 1 \ge 1$ with $p^{-1} + q^{-1} = 1$, we have

$$rs \le \frac{r^p}{p} + \frac{s^q}{q}$$

   This is called the Young's inequality.
   (c) Prove the Hölder's inequality.
   *(Hint: Consider the **normalized** case first, that is, the case where $\sum_{i=1}^{k} |x_i|^p = \sum_{i=1}^{k} |y_i|^q = 1$)*

4. This is a follow-up to Question 3. Let $k \in \mathbb{N}$ and $p \geq 1$. Let $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$. We can define $\|x\|_p := (\sum_{i=1}^k |x_i|^p)^{1/p}$, called the $p-$norm of x. We are going to show that $\|\cdot\|_p$ satisfy the triangle inequality.

   (a) Show that for all $\alpha \in \mathbb{R}$, we have $\|\alpha x\|_p = |\alpha| \|x\|_p$ for all $x \in \mathbb{R}^k$.

   (b) Show that $\|x\|_p = 0$ if and only if $x = 0$.

   (c) Using the Hölder's inequality, show that $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in \mathbb{R}^k$, that is, write $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_k)$, then we have

$$(\sum_{i=1}^k |x_i + y_i|^p)^{1/p} \leq (\sum_{i=1}^k |x_i|^p)^{1/p} + (\sum_{i=1}^k |y_i|^p)^{1/p}$$

   (Hint: Write $|x_i + y_i|^p = |x_i + y_i|^{p-1} |x_i + y_i|$ and try to see what the conjugate exponent of p is)

5. This is independent to Q4 but we are making use of the notations there. We are giving a proof for the triangle inequality of $p-$norms *without* using the Hölder's inequality for $p \geq 1$.

   (a) Let $x, y \in \mathbb{R}^k \backslash \{0\}$. Suppose $\|x\|_p + \|y\|_p = 1$. Show that there exists $X, Y \in \mathbb{R}^k$ with $\|X\|_p = \|Y\|_p = 1$ and $\lambda \in [0, 1]$ such that $x = \lambda X$ and $y = (1 - \lambda) Y$.

   (b) Show the triangle inequality for the case where $\|x\|_p + \|y\|_p = 1$.

   (c) Show the general triangle inequality.

   (d) (Reverse triangle inequality). Suppose $p \in (0, 1)$. Let $x, y \in \mathbb{R}^k$ be with non-negative entries, that is, we have $x_i, y_i \geq 0$ for all $i = 1, \cdots, k$. Then the reverse triangle inequality holds:

$$(\sum_{i=1}^k (x_i + y_i)^p)^{1/p} \geq (\sum_{i=1}^k x_i^p)^{1/p} + (\sum_{i=1}^k y_i^p)^{1/p}$$

6. Fix $k \in \mathbb{N}$. Let $x \in \mathbb{R}^k$. Let $p > 0$. We define $\|x\|_p := (\sum_{i=1}^k |x_i|^p)^{1/p}$.

   (a) Let $x \in (0, 1)$. Show that $x^p \leq x^q$ for all $p \geq q > 0$.

   (b) Show that for all $x \in \mathbb{R}^k$, we have $\|x\|_p \leq \|x\|_q$ for all $p \geq q > 0$.
   (Hint: Consider the case $\|x\|_q = 1$ first.)

3 Hölder's Continuity and a Nowhere Differentiable Example

Definition 3.1. Let $f : I \to \mathbb{R}$ be a function on some intervals. Let $\alpha \in (0, 1]$. Then we say that f is $\alpha-$Hölder continuous on I if there exists $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all $x, y \in I$. Note that $\alpha = 1$ is equivalent to the Lipschitz condition.

Quick Practice (Hard)

1. Let $\alpha > 1$ and $f : I \to \mathbb{R}$ be a function that is $\alpha-$Hölder continuous. Show that f is a constant function. This explains why we do not consider $\alpha > 1$ in the definition of Hölder continuity.

2. Let $\alpha \in (0, 1]$. Define $f(x) := x^\alpha$ on $[0, \infty)$.
   (a) Show that f is $\alpha-$ Hölder continuous on $[0, \infty)$
   (b) Show that f is not $\beta-$ Hölder continuous for all $\beta \neq \alpha$ on $[0, \infty)$ with $\beta \in (0, 1]$.
   (c) Show that f is $\beta-$ Hölder continuous for all $0 < \beta \leq \alpha$ on $A \subset [0, \infty)$ where A is bounded.

3. Let $f : I \to \mathbb{R}$ be a function over some interval I.
   (a) Show that f is uniformly continuous if f is $\alpha-$Hölder continuous for some $\alpha \in (0, 1]$.
   (b) Show that the converse is not true: there exists a uniformly continuous function that is *not* $\alpha-$Hölder continuous for all $\alpha \in (0, 1]$.
   (Hint: Consider $f : [0, 1/2] \to \mathbb{R}$ defined by $f(x) := 1/\log(x)$ for $x > 0$ with $f(0) := 0$)

4. Let $f : [0, 1] \to \mathbb{R}$ be a function. We say that f is locally $\alpha-$Hölder continuous ($\alpha \in (0, 1]$) at $x \in [0, 1]$ if there exists $r > 0$ such that $f\,|_{B_r(x)\cap[0,1]}$ is $\alpha-$Hölder continuous. We use the term *locally Lipschitz* for the case $\alpha = 1$.
   (a) Show that f is a Lipschitz function on $[0, 1]$ if and only if f is locally Lipschitz at x for all $x \in [0, 1]$.
   (b) Show that f is $\alpha-$Hölder continuous on $[0, 1]$ if and only if f is locally $\alpha-$Hölder continuous at x for all $x \in [0, 1]$

5. Let $f : [0, 1] \to \mathbb{R}$ be a function with $f(0) = f(1)$. Then we call $\bar{f} : \mathbb{R} \to \mathbb{R}$ the periodic extension of f if $\bar{f}(t + n) = f(t)$ for all $t \in [0, 1]$ and $n \in \mathbb{Z}$.
   (a) (20-21 2050 Rev. Exercise) Suppose f is continuous. Show that the periodic extension $\bar{f}$ is uniformly continuous.
   (b) Suppose f is $L-$Lipschitz. Show that the extension $\bar{f}$ is also $L-$Lipschitz. (f is $L-$Lipschitz if $|f(x) - f(y)| \leq L|x - y|$ for all possible x, y)
   (c) Suppose f is $\alpha-$Hölder continuous with $\alpha \in (0, 1]$. Is it true that the periodic extension $\bar{f}$ is also $\alpha-$Hölder continuous?

6. (Very challenging). Let $f : [0,1] \to \mathbb{R}$ be a function defined by $f(x) := \begin{cases} 0 & x = n/2,\ n \in \mathbb{Z} \textit{ is even} \\ 1 & x = n/2,\ n \in \mathbb{Z} \textit{ is odd} \end{cases}$ and extending linearly in between. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be the periodic extension of f to $\mathbb{R}$. It is not hard to see that f_1 is given by the same defining formula as f (with a different domain). Now define for all $k \in \mathbb{N}$ that $f_k(t) := 2^{-k+1} f_1(2^{k-1}t)$ for all $t \in \mathbb{R}$.

   (a) Show and convince yourself that for all $k \in \mathbb{N}$, we have

$$f_k(x) := \begin{cases} 0 & x = n/2^k,\ n \in \mathbb{Z} \textit{ is even} \\ 2^{-k+1} & x = n/2^k,\ n \in \mathbb{Z} \textit{ is odd} \end{cases}$$

   with other values given by the linear extensions between the defined dyadic points.

   (b) Show that for all $k \in \mathbb{N}$, f_k consists of straight lines of slopes ± 1 with horizontal length $1/2^k$. Furthermore, f_k are periodic with period $1/2^{k-1}$.

   (c) Fix $x \in \mathbb{R}$. Note that $f_k(x) \geq 0$ for all $k \in \mathbb{N}$. Define $F(x) := \sum_{i=1}^{\infty} f_i(x) := \lim_n \sum_{i=1}^{n} f_i(x) \in [0, \infty]$. Show that for all dyadic number $x \in \mathbb{R}$ (that is, $x = k/2^m$ for some $k \in \mathbb{Z}$ and $m \in \mathbb{N}$) we have that $F(x)$ is a finite sum. Moreover, show that $F(x) < \infty$ for all $x \in \mathbb{R}$.

   (d) Show that for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$, there exists $h_k \in \{\pm 1/2^{k+1}\}$ such that $|f_k(x + h_k) - f_k(x)| = |h_k|$ with additional facts that

      i. $|f_i(x + h_k) - f_i(x)| = |h_k|$ for all $i \leq k \in \mathbb{N}$.

      ii. $|f_i(x + h_k) - f_i(x)| = 0$ for all $i \geq k + 2 \in \mathbb{N}$.

   (e) Show that $F(x) := \sum_{i=1}^{\infty} f_i(x)$ is nowhere differentiable on $\mathbb{R}$, that is, F is not differentiable for all $x \in \mathbb{R}$.

   (f) (Hard) Show that F is not Lipschitz but F is $\alpha-$Hölder continuous for all $\alpha \in (0, 1)$.

7. This is a follow-up to Question 6. Using the same periodic function f_1 defined in the previous question, we define $g_k(t) := a^{k-1} f_1(2^{k-1}(t))$ for all $t \in \mathbb{R}$ where $a \in (0,1)$ with $2a > 1$. Define $G(t) := \sum_{i=1}^{\infty} g_i(t)$.

   (a) Show that G is well-defined,

   (b) Show that G is nowhere differentiable.

   (c) Show that G is α-Hölder continuous with $\alpha = -\log_2(a)$