1 Convergence of Functions

Definition 1.1. Let $D \subset \mathbb{R}$. Let $f_n : D \to \mathbb{R}$ be a sequence of functions. Let $f : D \to \mathbb{R}$. We say that

- $f_n \to f$ pointwise if $\lim_n f_n(x) = f(x)$ for all $x \in D$
- $f_n \to f$ uniformly if $\lim_n \sup_{x\in D} |f_n(x) - f(x)| = 0$. In other words, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.

Quick Practice

1. Let (f_n) be a sequence of functions on D.
   (a) Show that if $f_n \to f$ uniformly on D then $f_n \to f$ point-wise on D.
   (b) Show that $f_n \to f$ uniformly if and only if $f_n - f \to 0$ uniformly

2. Let (f_n) be a sequence of functions on $[0, 1]$.
   (a) Suppose $f_n \to f, g$ point-wise. Show that $f = g$ point-wise.
   (b) Suppose $f_n \to f, g$ uniformly. Show that $f = g$ point-wise.

3. Let (f_n) be a sequence functions on D. Suppose (f_n) converges uniformly.
   (a) Suppose (f_n) is a sequence of bounded function. Show that (f_n) is *uniformly bounded*, that is, there exists $M > 0$ such that $\sup_{x\in D} |f_n(x)| < M$ for all $n \in \mathbb{N}$.
   (b) Suppose (f_n) is a sequence of uniformly continuous functions. Show that (f_n) is *equi-continuous*, that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenver $x, y \in D$ are with $|x - y| < \delta$, that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$.

4. Suppose (f_n) is a sequence of functions over D that are continuous at $c \in D$. Further suppose $f_n \to f$ uniformly. Show that f is continuous at c.

5. Let (f_n) be a sequence of functions on D. Suppose (f_n) does not converge to 0 uniformly.
   (a) Show that there exists $\epsilon > 0$, a subsequence (f_{n_k}) of (f_n) and a sequence of points (x_n) in D such that $|f_{n_k}(x_k)| \geq \epsilon$ for all $k \in \mathbb{N}$.
   (b) Show that $\overline{\lim}_n \sup_{x\in D} |f_n(x)| > 0$.

6. For each of the following domains $D \subset \mathbb{R}$ and sequences of functions $f_n : D \to \mathbb{R}$,

   (a). Find its point-wise limit f.

   (b). Determine whether $f_n \to f$ uniformly.

   i. $f_n(x) := \dfrac{x}{x+n}, D = [0, t], t > 0$

   ii. $f_n(x) := \dfrac{x}{x+n}, D = [0, \infty)$

   iii. $f_n(x) := \dfrac{x^2 + nx}{n}, D = \mathbb{R}$

   iv. $f_n(x) := x^n, D = [0, t], t \in (0, 1)$

   v. $f_n(x) := x^n, D = [0, 1]$

   vi. $f_n(x) = x^{1+\frac{1}{2n+1}}, D = [-1, 1]$

   vii. $f_n(x) := \dfrac{1}{n(1+x^2)}, D = \mathbb{R}$

   viii. $f_n(x) := \begin{cases} 1 & x = 1, 1/2, \cdots, 1/n \\ 0 & otherwise \end{cases}, D = [0, 1]$

   ix. $f_n(x) := \begin{cases} x & x = 1, 1/2, \cdots, 1/n \\ 0 & otherwise \end{cases}, D = [0, 1]$

   x. Enumerate $\mathbb{Q} \cap [0, 1] = (q_n)$. Define $f_n(x) := \begin{cases} 1 & x = q_1, \cdots, q_n \\ 0 & otherwise \end{cases}, D = [0, 1]$

7. Let (f_n) be a sequence of functions on D.

   (a) (Cauchy Criteria). Show that f_n converges uniformly if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$, we have $|f_n(X) - f_m(x)| < \epsilon$ for all $x \in D$

   (b) (Weierstrass M-test). Define $S_n(x) := \sum_{i=1}^{n} f_i(x)$ for all $x \in D$ and $n \in \mathbb{N}$. Suppose there exists a sequence of summable positive real numbers (M_n) (that is $\sum M_n < \infty$) such that $\sup_{x \in D} |f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$. Show that S_n converges uniformly.