In the following, we are using the textbook notations that $L(f) := \underline{\int} f$ and $U(f) := \overline{\int} f$.

1 (P. 224 Q15)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $c > 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{x-c}^{x+c} f(t)dt$.

a. Show that g is differentiable on $\mathbb{R}$.

b. Find $g'(x)$.

Solution.

a. Define $F(x) := \int_0^x f(t)dt$ for all $t \in \mathbb{R}$. We first show that F is differentiable on $\mathbb{R}$ with $F' = f$. To this end, let $M > 0$. Note that f is continuous on $[-M, M]$ and $F(x) = \int_{-M}^x f(t)dt + \int_0^{-M} f(t)dt$. It follows that F is differentiable with $F' = f$ on $(-M, M)$ by applying FTC on its restriction on $[-M, M]$ (note that the second term is a constant). Since $M > 0$ is arbitrary, F is differentiable on on $\bigcup_M (-M, M) = \mathbb{R}$ with $F' = f$ there.
Next, note that from splitting domains, we have $g(x) = F(x + c) - F(x - c)$ for all $x \in \mathbb{R}$. It follows that g is differentiable as it is difference of two differentiable functions $F(x + c), F(x - c)$, which are in turn differentiable by the chain rule.

b. Since $F' = f$, it follows that $g'(x) = F'(x + c) - F'(x - c)$ for all $x \in \mathbb{R}$ by the chain rule.

Common Mistake. Note that the FTC is applied only on integrable functions defined on a *compact* interval and it only implies the differentiability of functions of the from $F(x) := \int_a^x f(t)dt$, in which the upper limit is the variable x and the lower limit is a constant. I expect you to show carefully (similar to the above solution that) in fact such F is differentiable on the unbounded $\mathbb{R}$ and that g is in turn differentiable by chain rule. Note that the intuition for the possible extension of differentiability of F from compact intervals to $\mathbb{R}$ is due to the fact that differentiability is a *local* behaviour.

2 (P. 233 Q3)**.** Let f, g be bounded functions on $I := [a, b]$. Suppose $f(x) \le g(x)$ for all $x \in I$, show that $L(f) \le L(g)$ and $U(f) \le U(g)$

Solution. Fix a partition $P \subset [a, b]$. It is clear that $U(f, P) \le U(g, P)$ and $L(f, P) \le L(g, P)$ as $f \le g$ point-wise everywhere. It follows from taking infimums and supremums respectively that $U(f) \le U(g)$ and $L(f) \le L(g)$.

3 (P. 233 Q8)**.** Let f be continuous on $I := [a, b]$. Suppose $f(x) \ge 0$ for all $x \in I$. Show that if $L(f) = 0$ then $f(x) = 0$ for all $x \in I$.

Solution. Suppose not. Then $f(c) > 0$ for some $c \in I$. Write $\epsilon := f(c)/2 > 0$. Then $f > \epsilon$ on some non-empty interval $J \subset I$ with $c \in J$. Write $J := [c, d] \subset [a, b]$ WLOG. Consider the parition $P := \{a, c, d, b\}$. Then $L(f, P) \ge \inf_{x\in[c,d]} f(x)(d - c) \ge \epsilon(d - c) > 0 = L(f)$, which is a contradiction (the first inequality made use of the non-negativity of f).

Common Mistake. The continuous assumption in this question is crucial. Failing to use the assumption means your solution is incorrect. In addition one should find an interval on which $f > \epsilon > 0$ for some $\epsilon > 0$. If you only consider $f > 0$ on some interval I, you should state also the Extreme Value Theorem to imply that $\inf f(I) = f(c) > 0$ for some $c \in I$. Otherwise, $f > 0$ on an interval I may not imply $\inf f(I) > 0$.