1 (P. 196 Q4)**.** Show that if $x > 0$, then we have

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x \tag{1}$$

Solution. Define $f(x) := \sqrt{1+x}$ for all $x > -1$. Then f is smooth on $(-1, \infty)$. Let $x > 0$. Then by Taylor's theorem, we have

$$f(x) - f(0) = f'(0)x + \frac{1}{2}f''(\xi)x^2$$

for some $\xi \in (0, x)$. Note that $f'(t) = \frac{1}{2}(1+t)^{-1/2}$ and $f''(t) = \frac{-1}{4}(1+t)^{-3/2}$ for all $t > -1$. It follows that we have $f'(0) = \frac{1}{2}$ and $f''(\xi) = \frac{-1}{4}(1+\xi)^{-3/2} \in [\frac{-1}{4}, 0]$ as $(1+\xi)^{-3/2} \in [0, 1]$ since $\xi > 0$. Hence, we have

$$\frac{-1}{8}x^2 \leq f(x) - f(0) - f'(0)x = \frac{1}{2}f''(\xi)x^2 \leq 0$$

The result follows by re-arranging the terms.

Remark. Parts of the can be obtained by considering the first and third derivatives as well.

2. Let $f(x) := e^x$ for all $x \in \mathbb{R}$. Show that the remainder term in Taylor's Theorem converges to 0 as $n \to \infty$ for all fixed $x_0, x \in \mathbb{R}$.

Solution. Consider $x_0 < x$ without loss of generality. Denote $R_n(x)$ the nth-order remainder term in Taylor's theorem with respect to x_0 for all $x \in \mathbb{R}$ and $n \in \mathbb{N}_{\geq 1}$, that is,

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i + R_n(x)$$

It follows that $R_n(x) = \frac{f^{(n)}(\xi_n)}{n!}(x - x_0)^n$ for some $\xi_n \in (x_0, x)$ for all $n \in \mathbb{N}$. We proceed to show that $\lim_n R_n(x) = 0$.

Method 1: Using $\epsilon - N$ definition. Note that $f^{(n)}(x) = f(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. In addition f is increasing. It follows that

$$0 \leq R_n(x) = \frac{f^{(n)}(\xi_n)}{n!}(x - x_0)^n = \frac{e^{\xi_n}}{n!}(x - x_0)^n \leq \frac{e^x}{n!}(x - x_0)^n$$

Write $a := x - x_0 > 0$. We claim that $\lim_n \frac{a^n}{n!} = 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > a$. Suppose $n \geq N$ and $n > a^{N+1}/\epsilon N!$. Then we have

$$\left|\frac{a^n}{n!}\right| = \frac{a^n}{n!} = \frac{a}{n} \cdots \frac{a}{N+1} \cdot \frac{a^N}{N!} \leq \frac{a}{n}\frac{a^N}{N!} < \epsilon$$

It follows from definition that $\lim_n \frac{a^n}{n!} = 0$. Hence, by sandwich theorem we have $\lim_n R_n(x) = 0$.

Method 2: Ratio Test. Note again that $f^{(n)}(x) = f(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Hence, we have

$$\frac{R_{n+1}(x)}{R_n(x)} = \frac{e^{\xi_{n+1}}}{e^{\xi_n}} \frac{(x - x_0)^{n+1}}{(x - x_0)^n} \frac{n!}{(n+1)!} = \frac{x - x_0}{n+1}e^{\xi_{n+1} - \xi_n} \leq \frac{x - x_0}{n+1}e^{x - x_0}$$

As $x - x_0$ is independent of n it follows that $\lim_n R_{n+1}(x)/R_n(x) = 0 < 1$. This implies that $\lim_n R_n(x) = 0$ (we can in fact deduce that $(R_n(x))$ is summable).

Common Mistake. It is important to note that the ξ obtained in Taylor's Theorem depends on x, x_0, n. Therefore to show that $\lim_n R_n(x) = 0$, one has to give a bound for e^{ξ_n}, or $e^{\xi_{n+1} - \xi_n}$ if the ratio test is used, so that the term is independent of n. Otherwise, you cannot take $n \to \infty$ with $\xi = \xi_n$ in hand. Many of you made this mistake.

3. Define $h(x) := \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

i. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

ii. Show that the remainder term in Taylor's Theorem for $x_0 = 0$ does not converge to 0 for all $x \neq 0$ as $n \to \infty$

Solution.

i. We first show that $\lim_{x\to 0} \frac{h(x)}{x^k} = 0$ for all $k \geq 1$. We proceed using induction. Write $f(x) := 1/x$ and $g(x) := e^{1/x^2}$. Note that $g'(x) \neq 0$ for all $x \neq 0$ and $\lim_{x\to 0} g(x) = \infty$. Furthermore $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{-x^{-2}}{-2x^{-3}g(x)} = \lim_{x\to 0} \frac{x}{2e^{1/x^2}} = 0$. By considering both 1-sided limits, it follows that the L'Hospital rule applies. Hence, $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{h(x)}{x} = 0$. Now suppose $\lim_{x\to 0} \frac{h(x)}{x^n}$ for all $n < k$. Write $f(x) := 1/x^k$ and $g(x) := e^{1/x^2}$. Then $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{-kx^{-k-1}}{-2x^{-3}g(x)} = \lim_{x\to 0} \frac{k}{2}\frac{h(x)}{x^{k-2}} = 0$ by induction hypothesis. Similarly a condition for L'Hospital Rule applies. Therefore $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{h(x)}{x^k} = 0$.

We now show that $\lim_{x\to 0} \frac{h^{(n)}(x)}{x^k} = 0$ for all $n, k \in \mathbb{N}$. By the above $\lim_{x\to 0} \frac{h(x)}{x^K} = 0$ for all $k \in \mathbb{N}$. Now fix $n \geq 1$. Suppose $\lim_{x\to 0} \frac{h^{(j)}(x)}{x^k}$ for all $j < n$ and $k \in \mathbb{N}$. Note that as $n \geq 1$, we have $h^{(n)}(x) = (h')^{(n-1)}(x)$ for all $x \neq 0$ while $h'(x) = -2x^{-3}h(x)$. It follows from the product rule (Leibniz's Rule) that for $x \neq 0$

$$h^{(n)}(x) = (h')^{(n-1)}(x) = (-2x^{-3}h(x))^{(n-1)} = \sum_{i=0}^{n-1} \binom{n-1}{i} (-2x^{-3})^{(i)} h^{(n-1-i)}(x)$$

By linearity and induction hypothesis, it follows that $\lim_{x\to 0} h^{(n)}(x)/x^k = 0$. Hence $\lim_{x\to 0} \frac{h^{(n)}(x)}{x^k} = 0$ for all $n, k \in \mathbb{N}$.

To the end, we conclude that $h^{(n)}(0) = 0$ by an induction argument: the case for $n = 0$ is clear. Let $n \geq 1$. Suppose $h^{(k)}(0) = 0$ for all $k < n$. Then we have by the induction hypothesis as well as previously proved results that

$$h^{(n)}(0) := \lim_{x\to 0} \frac{h^{(n-1)}(x) - h^{(n-1)}(0)}{x - 0} = \lim_{x\to 0} \frac{h^{(n-1)}(x)}{x} = 0$$

ii. Note that h is smooth on $\mathbb{R}$ by (i); we can apply Taylor's Theorem. Let $R_n(x)$ be the nth order remainder term for $x_0 = 0$ with $x \neq 0$. Then $h(x) = \sum_{i=1}^{n-1} \frac{h^{(i)}(0)}{i!} x^i + R_n(x)$. It follows from part (i) that $h(x) = R_n(x)$ for all $n \in N$ and $x \neq 0$. It is then clear that $\lim_n R_n(x) = h(x) \neq 0$ for all $x \neq 0$.

Common Mistake.

a). It is completely wrong to use $\lim_{x\to 0} \frac{h(x)}{x^k} = 0$ to deduce that $\lim_{x\to 0} \frac{h^{(k)}(x)}{x} = 0$ by the L'Hospital Rule. The aim of the L'Hospital Rule is to use derivatives to compute limits and require *in advance* that the limit of the derivative exists. Please revise the L'Hospital Rule.

b). Note that you cannot transform $\lim_{x\to 0} e^{-1/x^2} x^{-1}$ into $\lim_{y\to\infty} e^{-y^2} y$ as this is true for only $x \to 0^+$.

Remark.

i. This example is the standard example of a smooth function that does not admit Taylor's expansion at some points, that is, a smooth function that is not analytic. Similar behaviors do not occur in complex variables in which analyticity and complex differentiability (holomorphy) coincide.

ii. Some of you used the Taylor's theorem on e^{-1/x^2} to conclude that $e^{-1/x^2} \leq n!x^{2n}$ for $x \neq 0$ and for $n \in \mathbb{N}$. This is a good way to simplify the above proof.