1 (P. 179 Q7)**.** Show that for all $x > 1$, we have

$$\frac{x-1}{x} < \log x < x - 1$$

using the Mean Value Theorem.

Solution. Let $f(x) := \log(x)$ be defined for all $x > 0$. Then f is differentiable on $(0, \infty)$. Let $x > 1$. Then there exists $c \in (1, x)$ such that $f(x) - f(1) = (x-1)f'(c)$, that is, we have $\log x = (x-1)/c$ where $c \in (1, x)$. It is then clear that we can have

$$\frac{x-1}{x} < \log x = \frac{x-1}{c} < x - 1$$

as $c \in (1, x)$.

2 (P. 179 Q8)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that if $A := \lim_{x \to a^+} f'(x)$ exists then $f'(a) = A$.

Solution (See Tutorial 2 Exercise 3.1). Let $x \in (a, b)$. Then $f(x) - f(a) = f'(\xi(x))(x - a)$ where $\xi(x) \in (a, x)$ by MVT. Then $f'(\xi(x)) = \frac{f(x)-f(a)}{x-a}$. Now we consider $x \to a^+$ on both side. For the right expression, $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} = f'(a)$ as f is differentiable at a. For the left expression, we have to show that $\lim_{x \to a^+} f'(\xi(x)) = \lim_{x \to a^+} f'(x) = A$. We proceed with the $\epsilon - \delta$ definition. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $x - a < \delta$ would imply $|f'(x) - A| < \epsilon$. Note that if $x - a < \delta$ then $\xi(x) - a < x - a < \delta$. In particular, we have $|f'(\xi(x)) - A| < \epsilon$. It follows that $\lim_{x \to a^+} f'(\xi(x)) = A$.

Common Mistake. I do not accept something like since $x \to a^+$, $c \to a^+$. Therefore $A = \lim_{x \to a^+} f(x) = \lim_{x \to a^+} f(c)$ (where c is the point by MVT with a fixed x). At least you should address that $c = c(x)$ is a function of x and you are making use of the composition of continuity to claim that $\lim_{x \to a^+} f(c(x)) = \lim_{y \to a^+} f(y) = A$. To play safe, you are suggested to use an $\epsilon - \delta$ argument like the above.
Nonetheless, since your *intuition* is correct, some credit was still given.

3 (P. 179 Q19)**.** Let $f : I \to \mathbb{R}$ be a function where $I := (a, b)$. We say that f is *uniformly differentiable* on I if f is differentiable on I and for all $\epsilon > 0$, there exists $\delta < 0$ such that if $0 < |x - y| < \delta$ then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon$$

Show that if f is uniformly differentiable on I then f' is continuous on I.

Solution. We show that f' is uniformly continuous on I. This would imply f is continuous on I. Let $\epsilon > 0$. Then take $\delta > 0$ using the definition of *uniformly differentiability*. Now suppose $x, y \in I$ and $|x - y| < \delta$. Clearly we only have to consider the case that $x \neq y$. Then it follows from the definition of uniformly differentiability that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon \qquad \left| \frac{f(y) - f(x)}{y - x} - f'(y) \right| < \epsilon$$

It follows from the triangle inequality that

$$\begin{aligned} |f'(x) - f'(y)| &= \left| f'(x) - \frac{f(x) - f(y)}{x - y} + \frac{f(y) - f(x)}{y - x} - f'(y) \right| \\ &\leq \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| + \left| \frac{f(y) - f(x)}{y - x} - f'(y) \right| < 2\epsilon \end{aligned}$$

It follows from definition that f' is uniformly continuous on I. Therefore, f' is continuous on I

Common Mistake. It is disappointing that a number of you are showing the *uniform continuity* of f' without addressing it correctly. Remembering definitions is very important in studying Math and so quite a large portion of marks would be deducted if you fail to give correct definitions when needed.

Remark. In fact one can show that for a differentiable function on I, if f' is uniformly continuous then f is uniformly differentiable on I. Try this as an exercise.