MATH 2050A - HW 4 - Comments and Common Mistakes:

General Comments and Mistakes:

1. Well done all. I can clearly see your improvement writing the $\epsilon - N$ argument. (More than half of you got 9 or above this time). Please keep it up when you proceed to write the $\epsilon - \delta$ argument in the future.

2. As you have been more familiar with the $\epsilon - N$ arguments, it would be useful for you to note the following "simplifications" (try to prove them yourself):

   Proposition 0.1. *Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. Then the following are equivalent.*

   (a) (x_n) converges to x

   (b) There exists $r > 0$ such that for all $\epsilon \in (0, r)$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ when $n \geq N$

   (c) There exists $C > 0$ such that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < C\epsilon$ when $n \geq N$

3. A few of you still used results or definitions that have not been taught, which include but are not limited to, the Logarithmic function, the floor function, differentiation, the L'Hospital Rule, AM-GM Inequality. I sincerely invite you to provide proofs or verify the well-definedness of these results/definitions. Otherwise, I cannot help but deduct your marks.

   Remark. You may use the above results in *any* subsequent homework once you have proved in one homework (correctly and with only facts you have learnt from this course).

4. Quite a number of you have defined variables like the following:

   Let $y_n = 3^{\frac{1}{2n}}$.
   Since $3 \geq 1$, We have $3^{\frac{1}{2n}} \geq 1^{\frac{1}{2n}} = 1$
   So $3^{\frac{1}{2n}} = 1 + d_1$ for some $d_1 > 0$.

   Here d_1 should be a variable that depends on n. Write instead $y_n = 1 + d_n$ for some $d_n > 0$ for all $n \in \mathbb{N}$.Otherwise it would mean that the sequence is a constant, that is, $y_n = d_1$ for all $n \in \mathbb{N}$ for some $d_1 > 0$.
   The same problem occurs when defining subsequences.

5. Some of you used the Bolzano-Weierstrauss Theorem wrongly. I believe Professor Leung should have emphasized the power of this result (or any other results) but it doesn't mean you can always use those results.

   Since $1 < (X_n) \leq 3^{\frac{1}{2}}$, (X_n) is bounded.
   By the Bolzano-Weierstrass Theorem, (X_n) has a convergent subsequence.
   (X_n) converges to 1. why?

Good Work:

Question 1:

Show that $\left(1-(-1)^n+\frac{1}{n}\right)$ is divergent.

1. This student uses the Cauchy Criteria to do this question.

We proce this statement by contradiction.

Suppose that x_n is convergent.

According to the Cauchy Theorem, since x_n is a convergent seq, we have x_n is a Cauchy sequence.

Therefore, $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$ s.t. $|x_m-x_n|<\varepsilon$ as $m, n \geqslant N$.

choose $\varepsilon=1$, then $\exists N \in \mathbb{N}$ s.t.

$|x_m-x_n|<1$, $\forall m, n \geqslant N$.

Let m be an odd number, $m \geqslant N$, and let $n=m+1>N$.

Then $|x_m-x_n| = \left|(1+1+\frac{1}{m})-(1-1+\frac{1}{n})\right| = \left|2+\frac{1}{m}-\frac{1}{m+1}\right| = 2+\frac{1}{m}-\frac{1}{m+1}$

Since $m \in \mathbb{N}$, we have $\frac{1}{m}-\frac{1}{m+1}>0$.

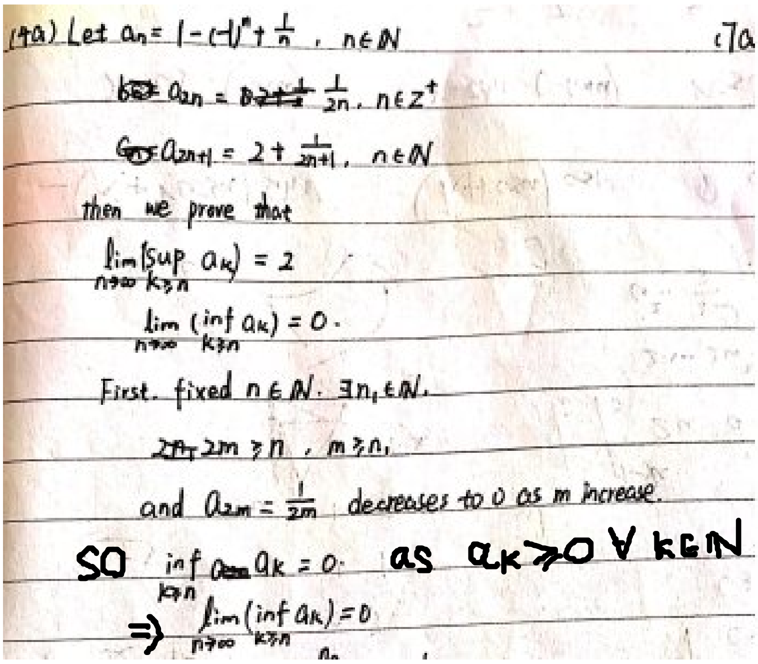
Therefore, $|x_m-x_n| = 2+\frac{1}{m}-\frac{1}{m+1}>2$.

However, according to the definition of Cauchy sequence, when $\varepsilon=1$, we have $|x_m-x_n|<1$.

Contradiction arises.

Hence, x_n is not convergent.

2. This student uses limsup and liminf to do this question.



(4a) Let $a_n = 1-(-1)^n + \frac{1}{n}$, $n \in \mathbb{N}$ (7a

~~Let~~ $a_{2n} = \frac{1}{2n}$, $n \in \mathbb{Z}^+$

$a_{2n+1} = 2 + \frac{1}{2n+1}$, $n \in \mathbb{N}$

then we prove that

$\lim_{n\to\infty}(\sup_{k\geq n} a_k) = 2$

$\lim_{n\to\infty}(\inf_{k\geq n} a_k) = 0$.

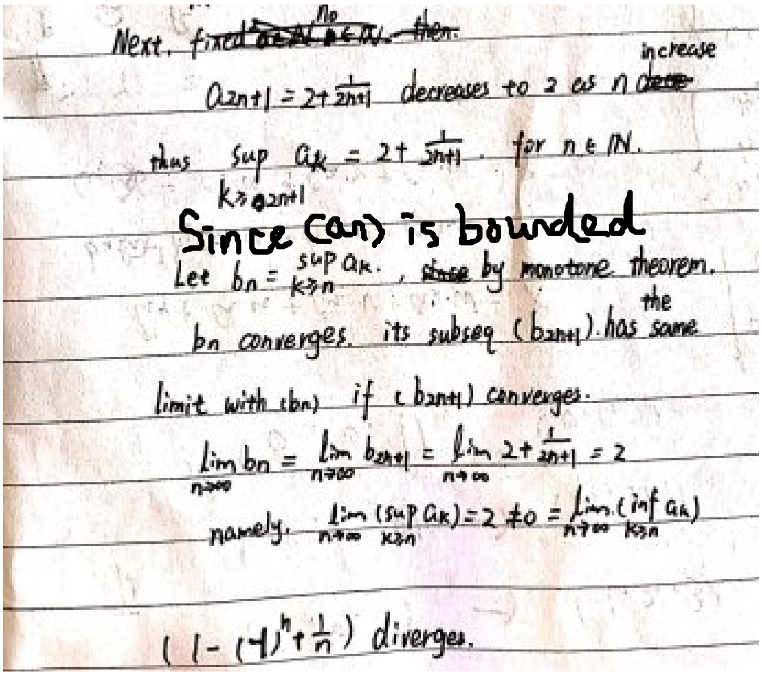
First. fixed $n \in \mathbb{N}$. $\exists n_1 \in \mathbb{N}$.

$2m \geq n$, $m \geq n_1$

and $a_{2m} = \frac{1}{2m}$ decreases to 0 as m increase.

SO $\inf_{k\geq n} a_k = 0$. as $a_k \geq 0\ \forall k \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty}(\inf_{k\geq n} a_k) = 0$



Next. fixed ~~...~~ then.

$a_{2n+1} = 2 + \frac{1}{2n+1}$ decreases to 2 as n increase

thus $\sup_{k\geq 2n+1} a_k = 2 + \frac{1}{2n+1}$. for $n \in \mathbb{N}$.

Since (a_n) is bounded

Let $b_n = \sup_{k\geq n} a_k$. , ~~since~~ by monotone theorem.

b_n converges. its subseq (b_{2n+1}) has the same limit with (b_n) if (b_{2n+1}) converges.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{2n+1} = \lim_{n\to\infty} 2 + \frac{1}{2n+1} = 2$

namely. $\lim_{n\to\infty}(\sup_{k\geq n} a_k) = 2 \neq 0 = \lim_{n\to\infty}(\inf_{k\geq n} a_k)$

$(1-(-1)^n + \frac{1}{n})$ diverges.

It is good that they apply what they have newly learnt.

Question 2:

Establish the convergence for the following sequence and find its limit.

$$\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)$$

1. This student pays attention to the well-definedness of a subsequence by making sure that the function $n \mapsto n^2$ is strictly increasing.

Let $x_n = \left(1+\frac{1}{n}\right)^n$ for $n \in \mathbb{N}$. Note that we have

$$\lim x_n = e.$$

Let $n \in \mathbb{N}$, note that $n^2 < n^2 + 2n + 1 = (n+1)^2$ and $n^2 \in \mathbb{N}$. It follows that $(n^2)_{n\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers. Hence, $(x_{n^2})_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. It follows that $(x_{n^2})_{n\in\mathbb{N}}$ is convergent and

$$\lim(x_{n^2}) = \lim\left(1+\frac{1}{n^2}\right)^{n^2} = \lim(x_n) = e.$$

Question 3:

Determine the limit of the following sequence. (It it possible that the limit does not exist).

$$\left((3n)^{\frac{1}{2n}}\right)$$

1. Basically, the key point of this question is to show $\lim_n n^{1/n} = 1$. This student has presented a very clear proof

Let $y_n = n^{\frac{1}{n}}$

Claim: (y_n) is convergent with limit $= 1$

Since $n^{\frac{1}{n}} > 1$ for $n > 1$,

$n^{\frac{1}{n}} = 1 + k_n$ for some $k_n > 0$ when $n > 1$,

Hence $n = (1 + k_n)^n$ for $n > 1$

By Binomial Theorem, if $n > 1$,

$$\begin{aligned} n &= C^n_0 + C^n_1 (k_n) + C^n_2 (k_n)^2 + \cdots + C^n_n (k_n)^n \\ &= 1 + nk_n + C^n_2 (k_n)^2 + \sum_{r=3}^{n} C^n_r (k_n)^r \\ &= 1 + nk_n + \tfrac{1}{2} n(n-1) k_n^2 + \sum_{r=3}^{n} C^n_r (k_n)^r \\ &\geq 1 + nk_n + \tfrac{1}{2} n(n-1) k_n^2 \qquad (\because C^n_r (k_n)^r \geq 0 \ \forall r \in [\![3, n]\!]) \\ &\geq 1 + \tfrac{1}{2} n(n-1) k_n^2 \qquad (\because nk_n \geq 0) \end{aligned}$$

$n - 1 \geq \frac{1}{2} n(n-1) k_n^2$

$k_n^2 \leq \frac{2}{n}$ (if $n > 1$)

Pick any $\varepsilon > 0$, $\frac{2}{\varepsilon^2} > 0$,

By the Archimedean Property, $\exists\, n \in \mathbb{N}$ s.t. $\frac{2}{\varepsilon^2} < n$

$\therefore 0 < \frac{2}{n} < \varepsilon^2$, $\sqrt{\frac{2}{n}} < \varepsilon$

$$\begin{aligned} |n^{\frac{1}{n}} - 1| = n^{\frac{1}{n}} - 1 = k_n &= \sqrt{k_n^2} \qquad (\because k_n > 0) \\ &\leq \sqrt{\tfrac{2}{n}} < \varepsilon \end{aligned}$$

$\therefore (n^{\frac{1}{n}})$ is convergent with $\lim y_n = \lim n^{\frac{1}{n}} = 1$