THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW2 Solution

1. (P.179 Q5)

   Following the hint, we consider $f(x) = x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ where $x \geq 1$. f is clearly continuous on $[1, +\infty)$ and differentiable on $(1, +\infty)$. Therefore, Mean Value Theorem (Theorem 6.2.4) is applicable on every finite subinterval $[1, d]$ for any $d > 1$.

   For any $x > 1$, $f'(x) = \dfrac{1}{n}(x^{\frac{1}{n}-1} - (x-1)^{\frac{1}{n}-1})$. Since $x > x-1 > 0$ and $\dfrac{1}{n} - 1 < 0$, $x^{\frac{1}{n}-1} < (x-1)^{\frac{1}{n}-1}$

   Therefore, $f'(x) < 0$ for any $x > 1$.

   Now given $a > b > 0$, consider $d = \dfrac{a}{b} > 1$. Applying Mean Value Theorem to f on $[1, d]$, there exists $c \in (1, d)$ such that

   $$f(d) - f(1) = f'(c)(d-1)$$

   Since $c > 1$, the above implies $f'(c) < 0$, and hence $f(d) - f(1) < 0$. Writing out the definitions explicitly, we have

   $$\begin{aligned} [(\frac{a}{b})^{\frac{1}{n}} - (\frac{a}{b} - 1)^{\frac{1}{n}}] - (1-0) &< 0 \\ a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}} &< b^{\frac{1}{n}} \end{aligned}$$

   Therefore, $a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}$.

   Remark: Many students tried to argue that $f'(x) < 0$ for $x \geq 1$, which is not true since f is actually not differentiable at $x = 1$. Even if $f'(x) < 0$ for all $x > 1$, one cannot immediately deduce that f is strictly decreasing on $(1, +\infty)$ without proving it (which is actually section 6.2 Q13). Finally, even if f is strictly decreasing on $(1, +\infty)$, it does not imply immediately that $f(1) > f(x)$ for all $x > 1$, since $1 \notin (1, +\infty)$. One has to use Mean Value Theorem to prove the final claim.

2. (P.179 Q7)
   For $x \geq 1$, let$f(x) = \ln x - x - 1$, $f(1) = 0$. For $x > 1$, by MVT, there exists $c_x \in (1, x)$ such that

   $$\frac{f(x) - f(1)}{x - 1} = f'(c_x).$$

   Check that for all $x > 1, f'(x) = \frac{1}{x} - 1 < 0$. Hence, we have proved the right one. Similarly, for the left one, For $x \geq 1$, let $g(x) = \frac{x-1}{x} - \ln x$, $g(1) = 0$. For $x > 1$, by MVT, there exists $q_x \in (1, x)$ such that

   $$\frac{g(x) - g(1)}{x - 1} = g'(q_x).$$

   Check that for all $x > 1, g'(x) = \frac{1}{x^2} - \frac{1}{x} =< 0$. Hence, we have also proved the left one.

3. (P.179 Q14)

   The proof is just a direct contradiction of Thm 6.2.12 Darboux's Theorem. (Intermediate value property of derivative well-defined on an interval.) If there exists distinct $x_1 < x_2 \in I$, and w.l.o.g., $f(x_1) < 0$ and $f(x_2) > 0$. by Thm 6.2.12, there exists $c \in (x_1, x_2) \subset I$ such that $f'(c) = 0$. Contradiction occurs.

4. (P.179 Q15)

   Since f' is bounded on I, there exists $M \in \mathbb{R}$ such that for all $w \in I$, $|f'(w)| \leq M$.

   To show f satisfies a Lipschitz condition on I, it suffices to show that there exists $L \in \mathbb{R}$ such that for all $x, y \in I$, $|f(x) - f(y)| \leq L|x - y|$

   We choose $L = M$ and claim that the above statement holds true: Given any $x, y \in I$,

   Case 1: $x = y$: then $|f(x) - f(y)| = 0 \leq 0 = L|x - y|$

   Case 2: $x < y$: Since I is an interval, $[x, y] \subseteq I$. Since f is differentiable on I, f is differentiable on $[x, y]$, and by Theorem 6.1.2 f is continuous on $[x, y]$; also f is differentiable on (x, y). Therefore, by Mean Value Theorem (Theorem 6.2.4), there exists $c \in (x, y)$ such that

   $$\frac{f(y) - f(x)}{y - x} = f'(c)$$

   Hence, $|f(y) - f(x)| = |f'(c)||y - x| \leq M|y - x|$.

   Case 3: $x > y$: interchanging the roles of x and y and adopt similar argument as in case 2 (i.e. replacing $[x, y]$ by $[y, x]$, etc.) , we have

   $$|f(x) - f(y)| \leq M|x - y|$$

   Therefore, for all $x, y \in I$, $|f(x) - f(y)| \leq L|x - y|$, and hence f satisfies a Lipschitz condition on I.

   Remark: Most students overlooked the case $x = y$. Although the argument is trivial, it is still essential as this is the only case where Mean Value Theorem is not applicable; also, some students combined case 2 and 3 together by saying "...there exists c between x and y...". This is ambiguous as it is not clear whether c could possibly be x or y by saying so (in other words, whether the "between" is inclusive and exclusive). It is better to split into cases for the sake of clarity.