MATH4010

Tutorial 9

Adjoint operator of an operator on Hilbert spaces

Let $T \in B(H)$. then $\exists ! \, T^* \in B(H)$ s.t.

$$(Tx, y) = (x, T^*y) \qquad \forall x, y \in H$$

- $\| T^* \| = \| T \|$
- $\| T^* T \| = \| T \|^2$
- $\operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$. $(\operatorname{Ker} T)^\perp = \overline{\operatorname{Im} T^*}$

Ex 1. $H = \mathbb{R}^n$. $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, $T: x \mapsto Ax = \sum_{j=1}^{n} a_{ij} x_j$

Then $T^* = ?$

$$\langle x, Ay \rangle = \sum_{ij} x_i a_{ij} y_j = \sum_{ij} y_j a_{ij} x_i = \langle A^T x, y \rangle$$

Ex 2. $H = L^2(\mathbb{R})$. $Tf(y) := \int k(x,y) f(x)\, dx$. $T^* = ?$

$$\begin{aligned}
\langle g, Tf \rangle &= \int \overline{g(y)} \int k(x,y) f(x)\, dx\, dy \\
&= \iint k(x,y) \overline{g(y)} f(x)\, dy\, dx \\
&= \int \overline{\int \overline{k(x,y)}\, g(y)\, dy}\; f(x)\, dx = \langle T^* g, f \rangle
\end{aligned}$$

$$T^* g(x) = \int \overline{k(x,y)}\, g(y)\, dy$$

Ex3. $K* : L^2 \to L^2$

$$f \mapsto (K*f)(x) = \int_{-\infty}^{\infty} K(x-y) f(y)\, dy \qquad K \in L^1$$

$$(K*)^* = ? \qquad \|K*f\|_{L^2} \le \|K\|_{L^1} \|f\|_{L^2}$$

suppose that $\check{K}(x) = K(-x)$.

$$\langle g, K*f \rangle = \iint g(x) K(x-y) f(y)\, dx\, dy$$

$$= \int \left(\int g(x) K(x-y)\, dx \right) f(y)\, dy$$

$$= \langle \check{K} * g, f \rangle$$

Ex4. If A is a linear self-adjoint operator, then

$$e^{iA} = \sum_{n=0}^{\infty} \frac{1}{n!} (iA)^n$$

is unitary, since

$$(e^{iA})^* = e^{-iA} = (e^{iA})^{-1}$$

Ex5. X, Y, Z Banach space. $A \in \mathcal{L}(X, Y)$, $A^{-1} \in \mathcal{L}(Y, X)$, $B \in \mathcal{L}(Y, Z)$

Then (1) $(A^*)^{-1}$ exist, $(A^*)^{-1} \in \mathcal{L}(X^*, Y^*)$, $(A^*)^{-1} = (A^{-1})^*$

(2) $(BA)^* = A^* B^*$

Proof. (1) $A^*(A^{-1})^* x^*(y) = A^*(x^*(A^{-1}y))$ $\quad \forall x^* \in X^*$

$= x^*(AA^{-1}y) = x^*(y)$ $\quad y \in Y$

$\Rightarrow A^*(A^{-1})^* = I$. ///

(2) $(BA)^* z^*(x) = z^*(BAx)$ $\quad \forall z^* \in Z^*$

$= B^* z^*(Ax) = A^* B^* z^*(x)$ $\quad x \in X$

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$\square$

Ex6. If A is a linear self-adjoint operator, and $\overline{\operatorname{Im} A} = H$

then there exists a $x \in H$. s.t. $Ax = y$ holds for any $y \in H$.

Proof. "Remark 14.4 in Lecture Notes" $\Rightarrow$ A is actually bounded.

It follows from the closed graph theorem that $\overline{\operatorname{Im} A} = \operatorname{Im} A$.

$\operatorname{Ker} A = \operatorname{Ker} A^* = (\operatorname{Im} A)^{\perp} = \{0\}$. Therefore, A is bijective.

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$\square$.