MATH4010 Solution 3

P127. Ex 8.10.17 Omitted

P129. Ex 8.14.1 Hint: $x_n = \left(\frac{3}{2}, (\frac{3}{2})^2, \dots, (\frac{3}{2})^n, 0, 0, \dots\right)$

$\|x_n\|_{\ell^\infty} \to \infty$. $\|Jx_n\|_{\ell^1} \le C$.

P239. Ex 11.26.9

Proof. (1) ① Denote the map $M^\perp \to (X/M)^*$ as T.

$$\phi \mapsto \psi, \quad \psi(x+M) := \phi(x)$$

If $x+M = y+M$, then $x-y \in M$. $\phi(x-y)=0$, $\phi(x)=\phi(y)$

Check whether the definition of functionals or operators on quotient spaces are well-defined.

And it is clear that ψ is linear, and $|\psi(x+M)| \le \|\phi\| \|x\|_X$

$$\Rightarrow |\psi(x+M)| \le \|\phi\| \cdot \inf_{y \in x+M} \|y\| = \|\phi\| \|x+M\|_{X/M}. \quad (1.1)$$

Therefore, $\psi \in (X/M)^*$. i.e. T is well defined.

② It is trivial that T is linear.

③ Suppose that $\phi \in M^\perp$ and $T\phi = 0$. Then for any $x \in X$, $\phi(x) = T\phi(x+M) = 0$. Therefore, $\phi = 0$. i.e. $\mathrm{Ker}(T) = \{0\}$.

Suppose that $\psi \in (X/M)^*$, then define $\phi(x) = \psi(x+M)$.

It is trivial that ϕ is linear. $\phi(x) = 0$ if $x \in M$.

$$|\phi(x)| \le \|\psi\| \|x+M\|_{X/M} \le \|\psi\| \|x\|_X. \quad (1.2)$$

So $\phi \in M^\perp$, and $T\phi = \psi$. Therefore T is bijective.

④ By the inequality (1.1) and (1.2), we have

$$\|T\phi\| = \|\phi\|.$$

So combining ①.②.③.④. T is an isomorphism between $M^\perp$ and $(X/M)^*$.

(2) ① Denote the map
$$X^*/M^\perp \longrightarrow M^* \quad \text{as } S.$$
$$\phi + M^\perp \longmapsto \phi|_M$$

Suppose that $\phi + M^\perp = \psi + M^\perp$, then $\phi - \psi \in M^\perp \Rightarrow (\phi-\psi)|_M = 0$ i.e. $\phi|_M = \psi|_M$. Meanwhile, $\phi|_M$ is linear and $|\phi|_M(x)| \leq \|\phi\|_{X^*}\|x\|$ (2.1)

Therefore, $\phi|_M \in M^*$ and thus S is well defined.

② It is trivial that S is linear.

③ Suppose that $\phi \in X^*$ and $S(\phi + M^\perp) = \phi|_M = 0$. Then by definition $\phi \in M^\perp$, $\phi + M^\perp = 0 + M^\perp$, i.e. $\operatorname{Ker}(S) = \{0 + M^\perp\}$

Suppose that $\psi \in M^*$. Then by Hahn-Banach theorem, $\exists \phi \in X^*$ s.t. $\phi(x) = \psi(x)$, $\forall x \in M$ and $\|\phi\|_{X^*} = \|\psi\|_{M^*}$. (2.2) Hence,

$S(\phi + M^\perp) = \psi$. Then S is bijective.

④ It follows from (2.1) that

$$|\phi|_M(x)| \leq \inf_{\varphi \in \phi + M^\perp} \|\varphi\|_{X^*}\|x\| = \|\phi + M^\perp\|_{X^*/M^\perp}\|x\|$$

Meanwhile (2.2) implies $\quad \|\phi\|_{X^*} = \|\psi\|_{M^*} \geq \|\phi + M^\perp\|_{X^*/M^\perp}$

Therefore, $\|\phi + M^\perp\|_{X^*/M^\perp} = \|\phi|_M\|_{M^*}$. Combining ①.②.③.④, ... $\square$