Absolute Convergence and Conditional Convergence

We have discussed the notation of the convergence of series of numbers. Let's talk about a stronger version of convergence.

Definition (c.f. Definition 9.1.1)**.** Let (x_n) be a sequence of real numbers. The series $\sum x_n$ is said to *converge absolutely* if the series $\sum |x_n|$ is convergent. It is said to *converge conditionally* if it is convergent but not absolutely convergent.

The following theorem is a quick deduction from the **Cauchy Criterion of Series** and the **triangle inequality**.

Theorem (c.f. Theorem 9.1.2)**.** *A series must be convergent if it is absolutely convergent.*

Example 1. Observe the following examples.

- Every convergent series with non-negative terms is absolutely convergent. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}.$$

- The series $\sum \dfrac{\cos n}{n^2}$ is absolutely convergent from the discussion in the previous tutorial.

- The (alternating harmonic) series $\sum (-1)^{n+1}/n$ is conditionally convergent. Since the harmonic series is divergent, it suffices to show that this series is convergent. Note that

$$s_{2n} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$
$$s_{2n+1} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$$

Thus (s_{2n}) is an increasing sequence and (s_{2n+1}) is a decreasing sequence with

$$0 < s_{2n} < s_{2n+1} < 1.$$

By **Monotone Convergence Theorem**, both subsequences are convergent. Moreover, they converge to the same value α because

$$s_{2n+1} = s_{2n} + \frac{1}{2n+1}.$$

Then for any $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|s_{2n} - \alpha| < \varepsilon, \quad \forall n \geq N_1 \quad \text{and} \quad |s_{2n+1} - \alpha| < \varepsilon, \quad \forall n \geq N_2. \tag{1}$$

Take $N = \max\{2N_1, 2N_2 + 1\}$. Then whenever $n \geq N$, by (1), we have

$$\frac{n}{2} \geq \frac{N}{2} \geq N_1 \Longrightarrow |s_n - \alpha| = |s_{2(\frac{n}{2})} - \alpha| < \varepsilon \qquad \text{if } n \text{ is even;}$$
$$\frac{n-1}{2} \geq \frac{N-1}{2} \geq N_2 \Longrightarrow |s_n - \alpha| = |s_{2(\frac{n-1}{2})+1} - \alpha| < \varepsilon \qquad \text{if } n \text{ is odd.}$$

Hence $|s_n - \alpha| < \varepsilon$ no matter n is even or odd. It follows that the alternating harmonic series also converge to α.

Rearrangement Theorem (c.f. 9.1.5)**.** *Let $\sum x_n$ be an absolutely convergent series. Then for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum x_{\sigma(n)}$ is also convergent and*

$$\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n=1}^{\infty} x_n.$$

Remark. This convergence property is called **unconditional convergence**. The rearrangement theorem says that unconditional convergence is implied by absolute convergence.

Tests of Absolute Convergence

Last time, we have discussed some test for convergence. Let's recall the **Comparison Test** and see some more tests of absolute convergence.

Comparison Test (c.f. 3.7.7)**.** *Let (x_n) and (y_n) be sequences of real numbers. Suppose there exists $K \in \mathbb{N}$ such that*

$$0 \le x_n \le y_n, \quad \forall n \ge K.$$

Then

(a) the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.

(b) the divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

Root Test (c.f. 9.2.2)**.** *Let (x_n) be a sequence real numbers.*

(a) If there exists $r < 1$ and $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} \le r, \quad \forall n \ge K,$$

then $\sum x_n$ is absolutely convergent.

(b) If there exists $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} \ge 1, \quad \forall n \ge K,$$

then $\sum x_n$ is divergent.

Ratio Test (c.f. 9.2.4)**.** *Let (x_n) be a sequence of non-zero real numbers.*

(a) If there exists $r < 1$ and $K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \le r, \quad \forall n \ge K,$$

then $\sum x_n$ is absolutely convergent.

(b) If there exists $K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \ge 1, \quad \forall n \ge K,$$

then $\sum x_n$ is divergent.

Integral Test (c.f. 9.2.6)**.** *Let $f : [1, \infty) \to \mathbb{R}$ be a continuous, decreasing, positive function. Then $\sum f(n)$ is convergent if and only if the improper integral*

$$\int_1^\infty f(x)dx$$

exists. In this case, the limit is given by

$$\sum_{n=1}^\infty f(n) = \int_1^\infty f(x)dx.$$

Example 2 (c.f. Section 9.2, Ex.9)**.** Let $0 < a < 1$ and consider the series

$$a^2 + a + a^4 + a^3 + \cdots + a^{2n} + a^{2n-1} + \cdots .$$

Show that the **Root Test** applies but the **Ratio Test** does not apply.

Remark. Notice that the series is a rearrangement of the absolutely convergent geometric series so it must be convergent.

Solution. To apply the **Root Test**, we need to estimate $|x_n|^{1/n}$ for large n's.
For even $n = 2k$,

$$|x_n|^{1/n} = |x_{2k}|^{1/2k} = |a^{2k-1}|^{1/2k} = a^{1-1/2k} = a^{1-1/n}.$$

For odd $n = 2k + 1$,

$$|x_n|^{1/n} = |x_{2k+1}|^{1/(2k+1)} = |a^{2k+2}|^{1/(2k+1)} = a^{1+1/(2k+1)} = a^{1+1/n}.$$

In both cases, we have $|x_n|^{1/n} = a^{1\pm1/n}$. Hence

$$\lim_{n\to\infty} |x_n|^{1/n} = \lim_{n\to\infty} a^{1\pm1/n} = a < 1.$$

Therefore we see that the **Root Test** applies.

To apply the **Ratio Test**, we need to estimate $\left|\frac{x_{n+1}}{x_n}\right|$ for large n's.
For even $n = 2k$,

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{x_{2k+1}}{x_{2k}}\right| = \frac{a^{2k+2}}{a^{2k-1}} = a^3 < 1.$$

For odd $n = 2k + 1$,

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{x_{2k+2}}{x_{2k+1}}\right| = \frac{a^{2k+1}}{a^{2k+2}} = \frac{1}{a} \geq 1.$$

Hence the sequence $(|x_{n+1}/x_n|)$ is alternating between a^3 and a^{-1}, which lie in opposite sides of 1. Therefore we see that the **Ratio Test** does not apply.

Example 3 (c.f. Section 9.2, Ex.2, 3, 4 & 7)**.** Determine the convergence of the following series.

(a) $\sum_{n=1}^{\infty} n^n e^{-n}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(c) $\sum_{n=2}^{\infty} (\ln n)^{-\ln n}$

(d) $\sum_{n=2}^{\infty} (n \ln n)^{-1}$

(e) $\sum_{n=1}^{\infty} n! e^{-n^2}$

(f) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

Solution. Let's check the convergence of the series using suitable tests.

(a) We use the **Root Test** here. Note that

$$|x_n|^{1/n} = |n^n e^{-n}|^{1/n} = \frac{n}{e} \geq 1, \quad \forall n \geq 3.$$

Hence the series is **divergent**.

(b) We use the **Ratio Test** here. Note that

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n}.$$

Therefore we have

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e} < 1.$$

Hence the series is **convergent**.

(c) We use the **Comparison Test** here. Note that

$$\ln(x_n) = -\ln n \ln(\ln n) \leq -2\ln n, \quad \forall n \geq 2000.$$

(Here we want $\ln(\ln n) \geq 2$. i.e., $n \geq e^{e^2} \approx 1618.17$.) Hence we have

$$0 \leq x_n \leq \frac{1}{n^2}, \quad \forall n \geq 2000.$$

Since $\sum 1/n^2$ is convergent, the series is also **convergent**.

(d) We use the **Integral Test** here. Consider the function $f : [2, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x \ln x}.$$

Then f is a continuous, decreasing, positive function with $f(n) = x_n$. Also, the improper integral (if it exists) is given by

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_2^{\infty} \frac{1}{\ln x} d(\ln x) = \ln(\ln x)\Big|_2^{\infty}.$$

We can see that the improper integral does not exist, therefore the series is **divergent**.

(e) We use the **Ratio Test** here. Note that

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)!e^{-(n+1)^2}}{n!e^{-n^2}} = \frac{(n+1)!}{n!} \cdot \frac{e^{n^2}}{e^{(n+1)^2}} = \frac{n+1}{e^{2n+1}}.$$

Apply L'Hospital's Rule, we have

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} = \lim_{n\to\infty}\frac{1}{2e^{2n+1}} = 0 < 1.$$

Hence the series is **convergent**.

(f) We use the n**-th Term Test** here. Note that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty}\frac{(-1)^{2n}\cdot 2n}{2n+1} = 1 \neq 0.$$

Since we have found a subsequence of (x_n) that does not converge to 0, (x_n) must not converge to 0. Hence the series is **divergent**.